

*This paper has considered a task to expand the scope of application of fuzzy mathematics methods, which is important from a theoretical and practical point of view. A case was examined where the parameters of fuzzy numbers' membership functions are also fuzzy numbers with their membership functions. The resulting bifuzziness does not make it possible to implement the standard procedure of building a membership function. At the same time, there are difficulties in performing arithmetic and other operations on fuzzy numbers of the second order, which practically excludes the possibility of solving many practical problems. A computational procedure for calculating the membership functions of such bifuzzy numbers has been proposed, based on the universal principle of generalization and rules for operating on fuzzy numbers. A particular case was tackled where the original fuzzy number's membership function contains a single fuzzy parameter. It is this particular case that more often occurs in practice. It has been shown that the correct description of the original fuzzy number, in this case, involves a family of membership functions, rather than one. The simplicity of the proposed and reported analytical method for calculating a family of membership functions of a bifuzzy quantity significantly expands the range of adequate analytical description of the behavior of systems under the conditions of multi-level uncertainty. A procedure of constructing the membership functions of bifuzzy numbers with the finite and infinite carrier has been considered. The method is illustrated by solving the examples of using the developed method for fuzzy numbers with the finite and infinite carrier. It is clear from these examples that the complexity of analytic description of membership functions with hierarchical uncertainty is growing rapidly with the increasing number of parameters for the original fuzzy number's membership function, which are also set in a fuzzy fashion. Possible approaches to overcoming emerging difficulties have been described*

**Keywords:** *fuzzy mathematics, membership function of type 2 fuzzy numbers, construction rules*

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# DEVISING A METHOD FOR FINDING A FAMILY OF MEMBERSHIP FUNCTIONS TO BIFUZZY QUANTITIES

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## 1. Introduction

Fuzzy mathematics [1, 2] offers an effective toolkit for building models of systems that operate in uncertain environments. The increasing use of techniques from this section of mathematics is due to the ease of determining the values directly describing the uncertainty of the environment and the system. In addition, a fuzzy set theory, unlike probability theory, is not rigidly axiomatized. As one knows, the basic concept of probability theory is a probabilistic experiment. The mathematical model of the probability experiment is based on the Borel sigma-algebra of the numerical set of probabilistic experiments. The fundamental position of this algebra is that each probabilistic event is matched with the number – the probability of this event. The set of these

events should satisfy the rigid system of axioms [5]. However, numerous studies in recent years [6, 7] have drawn attention to the fact that the totality of observations of real objects operating in uncertain conditions does not fully meet the requirements of the basic axioms of probability theory. This fact naturally calls into question the correctness of the results from the theoretically probabilistic analysis of them. The most important feature of a fuzzy set theory is the structural easing of the requirements for the basic concept of this theory – the membership function of a fuzzy number. In particular, there is no need to comply with the normalization of this function. An analytical description of membership functions is formed on the basis of statistical treatment of actual data. However, this procedure is fundamentally impossible for fuzzy second-order numbers. The reason for the

problems here is the hierarchical nature of the description of the membership functions of fuzzy second-order numbers. That completely excludes the possibility of practical use of such numbers. At the same time, problems arise when solving numerous problems of system analysis, as well as their structural and parametric synthesis. It is clear, however, that the use of fuzzy second-order numbers improves the adequacy of an analytical description of the actual uncertainty of the original data. That makes the task of developing a technique to build a membership function for bifuzzy numbers undoubtedly relevant.

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## 2. Literature review and problem statement

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The required and sufficient way to describe fuzzy quantities is to assign a membership function. This problem, as it is known [1, 2], is solved as follows: choose the type of appropriate function and set its parameters, or they are evaluated statistically. For example, the triangular fuzzy number  $x$  is determined by the following membership function:

$$\mu_x(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{c-a}, & a \leq x < c, \\ \frac{b-x}{b-c}, & c \leq x \leq b, \\ 0, & x > b. \end{cases} \quad (1)$$

The shape of this membership function is shown in Fig. 1.

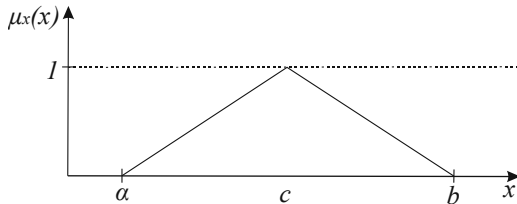


Fig. 1. Triangular fuzzy number membership function

The different types of membership functions (triangle, trapezoidal,  $(L-R)$  type, etc.) are often used in problem-solving to provide an adequate description of the state of the environment and system. In practice, however, there are often practical tasks that require more subtle and complex approaches to building models of actual objects.

Consider, for example, the task of describing the demand for a certain product. Suppose this demand is described by fuzzy triangular number (1). However, when one defines the parameters  $(a, b, c)$  on the appropriate membership function, it turns out that these parameters cannot be specified accurately. If fuzzy numbers are also used to describe these parameters, the original fuzzy number  $x$  is bifuzzy. Such fuzzy numbers were introduced in [1]; they are termed fuzzy second-order numbers. The bifuzzy numbers are mentioned in [2, 8], and, in more detail, in [9–13]. Those works give a definition of such numbers, provide a meaningful interpretation of the phenomenon of hierarchical uncertainty, and offer an analytical description of the membership functions to fuzzy numbers of the second order. This description is based on descriptions of the membership function to the fuzzy number and the membership functions to its fuzzy parameters. However, no formal techniques for operating on them have been defined. The authors of [14] and, later, of [15] considered a general approach to the use of fuzzy second-order numbers. The actual

tasks that involve these numbers are reviewed in [16]. A specific decision-making task for fuzzy numbers of the second type is considered and solved in [17]. Note, however, that the description of uncertainty, in that case, involved the simplest variant, that is interval numbers of the  $(LR)$ -type. In [18], using numbers of the same type, the statement of a control observation task was given, which was solved in [19]. The lack of adequacy in the descriptions of actual uncertainty of the original data in terms of the second-order interval numbers is noted in [20]. There, the problem of finding a membership function for fuzzy second-type numbers was properly set in a general statement. In [21], for a specific problem of regression analysis, it was proposed to use the approximation of the membership function of a fuzzy number of the second order in the form of two triangular functions. However, the level of accuracy of the solution was not discussed. Given this, we formulate the purpose and objectives of this study.

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## 3. The aim and objectives of the study

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The aim of this study is to devise a method for finding an analytical description of the membership functions to bifuzzy numbers.

To accomplish the aim, the following tasks have been set:

- to devise a basic technique to build the membership functions of bifuzzy numbers;
- to implement the basic technique for bifuzzy numbers with the finite and infinite carrier.

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## 4. The study materials and methods

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There is no technique to build membership functions for bifuzzy numbers. The fundamental reason is a separate description of the membership functions to the fuzzy number and the membership functions to its fuzzy parameters. This feature of the original descriptions of bifuzzy numbers does not make it possible to apply known theoretical methods to construct such functions. In this regard, in accordance with the aim of this study, we consider the following unconventional approach to the tasks formulated. The original problem data are to be interpreted as follows. There is a fuzzy number  $a$  with the membership function  $\mu_a(a)$ . This fuzzy number is being transformed using the function  $u = \mu_x(x, a)$ . It is clear that each value of the fuzzy number  $x$ , due to the fuzziness of  $a$ , is matched with its membership function to the arising bifuzzy number  $u$ . The task of finding these membership functions is to be solved as follows. In accordance with the generalization principle [1] and the rules of unary operations over fuzzy numbers, we find  $a = \mu_x^{-1}(x, u)$ . The desired membership function takes the following form:

$$\mu(u) = \mu_a(\mu_x^{-1}(x, u)).$$

The technique of calculating the membership function to a bifuzzy quantity  $u$  is illustrated using the simplest example. Let the fuzzy number  $x$ 's membership function be set in the interval  $[a, c]$  and represented as follows:

$$\mu(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{c-a}, & a \leq x \leq c, \\ 0, & x > c. \end{cases} \quad (2)$$

Assume that  $a$  parameter of membership function (2) is itself a fuzzy quantity with the following membership function:

$$\mu_a(a) = \begin{cases} 0, & a < a_0, \\ \frac{a-a_0}{a_1-a_0}, & a_0 \leq a \leq a_1, \\ 0, & a > a_1. \end{cases} \quad (3)$$

Find a membership function to the bifuzzy quantity  $u$ . Assume that the fuzzy number  $a$  with membership function (3) underwent a transformation according to (2), which yielded the following number:

$$u = \frac{x-a}{c-a}, \quad x \in [a, c]. \quad (4)$$

The following technique is used to find the membership function to the resulting fuzzy number  $u$ . From (4), we obtain:

$$a = \frac{uc-x}{u-1}.$$

Then, considering (3),

$$\mu(u) = \mu_a(x, u).$$

The analytical expression for  $\mu(u)$  depends on the choice of value for  $x$ . This is how the fundamental difference between the ordinary and bi-fuzzy fuzzy numbers is manifested. The membership function  $\mu(x)$  of the ordinary fuzzy number  $x$  assigns to each clear  $x$  value a clear number  $\mu(x)$ , which determines the level of membership of the chosen  $x$  value to its carrier. On the contrary, the specific value of the bifuzzy number  $x$  is assigned with the fuzzy number  $u$  with its membership function, which depends on the fuzzy parameter  $a$  that generates bifuzziness. This creates a family of membership functions, each representative of which is the membership function to a fuzzy number  $u$  that depends on the values of  $x$  and  $a$ .

### 5. Devising a technique to find a family of membership functions to bifuzzy numbers

Consider a technique to build the membership function to the bifuzzy number  $x$  with a fuzzy parameter  $a$ .

Let  $x > a$ . Then,  $a \in [a_0, a_1]$ . Hence:

$$\mu_1(u) = \begin{cases} 0, & \frac{uc-x}{u-1} < a_0, \\ \frac{\frac{uc-x}{u-1} - a_0}{a_1 - a_0}, & a_0 \leq \frac{uc-x}{u-1} \leq a_1, \\ 0, & \frac{uc-x}{u-1} > a_1. \end{cases} \quad (5)$$

Transform the resulting expression. We obtain:

$$\begin{aligned} \frac{uc-x}{u-1} - a_0 &= \frac{uc-x-ua_0+a_0}{u-1} = \frac{uc-c+c-x-a_0(u-1)}{u-1} = \\ &= \frac{c(u-1)-a_0(u-1)+c-x}{u-1} = c-a_0 + \frac{c-x}{u-1}. \end{aligned}$$

In addition, it follows from  $\frac{uc-x}{u-1} < a_0$ :

$$\frac{uc-x}{u-1} - a_0 = \frac{uc-x-ua_0+a_0}{u-1} = \frac{u(c-a_0)-(x-a_0)}{u-1} < 0,$$

hence

$$u(c-a_0) > (x-a_0), \quad u > \frac{x-a_0}{c-a_0}.$$

By analogy: from  $\frac{uc-x}{u-1} \geq a_0$  it follows  $u \leq \frac{x-a_0}{c-a_0}$ , from  $\frac{uc-x}{u-1} \geq a_1$  it follows  $u \geq \frac{x-a_1}{c-a_1}$ , from  $\frac{uc-x}{u-1} > a_1$  it follows  $u < \frac{x-a_1}{c-a_1}$ .

Now, considering these ratios, write (5) as follows:

$$\mu_1(u) = \begin{cases} 0, & u < \frac{x-a_1}{c-a_1}, \\ \frac{c-a_0 + \frac{c-x}{u-1}}{a_1-a_0}, & \frac{x-a_1}{c-a_1} \leq u \leq \frac{x-a_0}{c-a_0}, \\ 0, & u > \frac{x-a_0}{c-a_0}. \end{cases} \quad (6)$$

Calculate the values of membership function (6) at the extreme points of the fuzzy number  $u$ 's carrier.

It follows from (6):

$$\begin{aligned} \mu_1\left(\frac{x-a_1}{c-a_1}\right) &= \frac{c-a_1}{a_1-a_0} + \frac{c-x}{(a_1-a_0)\left(\frac{x-a_1}{c-a_1}-1\right)} = \\ &= \frac{1}{a_1-a_0} + \left[ c-a_0 + \frac{(c-x)(c-a_1)}{(x-a_1-c+a_1)} \right] = \\ &= \frac{1}{a_1-a_0}(c-a_0-c+a_1) = 1, \end{aligned}$$

$$\begin{aligned} \mu_1\left(\frac{x-a_0}{c-a_0}\right) &= \frac{c-a_0}{a_1-a_0} + \frac{c-x}{(a_1-a_0)\left(\frac{x-a_0}{c-a_0}-1\right)} = \\ &= \frac{1}{a_1-a_0} + \left[ c-a_0 + \frac{(c-x)(c-a_0)}{(x-a_0-c+a_0)} \right] = \\ &= \frac{1}{a_1-a_0}(c-a_0-c+a_1) = 0. \end{aligned}$$

Find a fuzzy number  $u$ 's membership function for some specific values  $x \in [a_1, c]$ .

Assume  $x = a_1$ . Then, for this value  $x$ :

$$\mu_1(u) = \begin{cases} 0, & u < 0, \\ \frac{c-a_0 + \frac{c-a_1}{u-1}}{a_1-a_0} = \frac{(c-a_0)u - (a_1-a_0)}{(u-1)(a_1-a_0)}, & 0 \leq u \leq \frac{a_1-a_0}{c-a_0}, \\ 0, & u > \frac{a_1-a_0}{c-a_0}. \end{cases}$$

The chart of this membership function takes the form shown in Fig. 2.

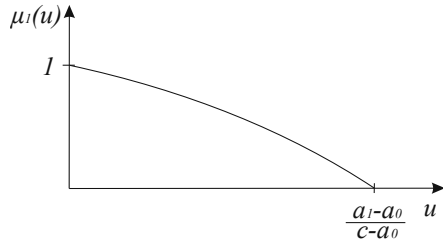


Fig. 2. Chart  $\mu_1(u)$  for  $x = a_1$

Now assume  $x = \frac{a_1 + c}{2}$ . Then, since:

$$\frac{\frac{a_1 + c}{2} - a_1}{c - a_1} = \frac{1}{2},$$

$$\frac{\frac{a_1 + c}{2} - a_0}{c - a_0} = \frac{c_0 - a_0 + a_1 - a_0}{2(c - a_0)} = \frac{1}{2} + \frac{a_1 - a_0}{2(c - a_0)},$$

$$\frac{c - a_0 + \frac{a_1 + c}{2}}{a_1 - a_0} = \frac{2(c - a_0)u - (c - a_0) - (a_1 - a_0)}{(u - 1)(a_1 - a_0)},$$

then

$$\mu_1(u) = \begin{cases} 0, u < \frac{1}{2}, \\ \frac{2(c - a_0)(u - 1) + (c - a_1)}{2(a_1 - a_0)(u - 1)}, \\ \frac{1}{2} \leq u \leq \frac{1}{2} + \frac{a_1 - a_0}{2(c - a_0)}, \\ 0, u > \frac{1}{2} + \frac{a_1 - a_0}{2(c - a_0)}. \end{cases}$$

The chart of this membership function is shown in Fig. 3.

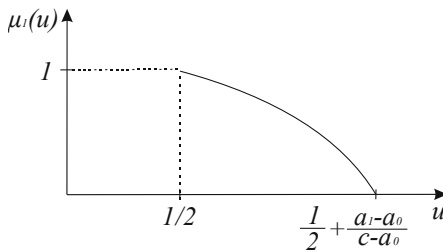


Fig. 3. Chart  $\mu_1(u)$  for  $x = \frac{a_1 + c}{2}$

Now assume  $x \in [a_1, a_0]$ . Then  $x \in [a_0, x]$ . In this case,

$$\mu_2(u) = \begin{cases} 0, \frac{uc - x}{u - 1} < a_0, \\ \frac{uc - x}{u - 1} - a_0, \\ \frac{uc - x}{u - 1} - a_0, \\ x - a_0, \\ a_0 \leq \frac{uc - x}{u - 1} \leq x, \\ 0, \frac{uc - x}{u - 1} > x. \end{cases}$$

Transform the expression for  $\mu_2(u)$  considering  $u < 1$ . We obtain:

$$\frac{uc - x}{u - 1} - a_0 = \frac{(x - a_0) - u(c - a_0)}{(1 - u)(x - a_0)}.$$

In addition, it follows from the inequality  $\frac{uc - x}{u - 1} \leq x$  that  $u \geq 0$ ; it follows from  $\frac{uc - x}{u - 1} > x$  that  $u < 0$ . Then the  $\mu_2(u)$  membership function takes the following form:

$$\mu_2(u) = \begin{cases} 0, u < 0, \\ \frac{(x - a_0) - u(c - a_0)}{(1 - u)(x - a_0)}, \\ 0 \leq u \leq \frac{x - a_0}{c - a_0}, \\ 0, u > \frac{x - a_0}{c - a_0}. \end{cases} \quad (7)$$

In this case, the values of  $\mu_2(u)$  at the carrier's extreme points are equal to:

$$\mu_2(0) = 1, \quad \mu_2\left(\frac{x - a_0}{c - a_0}\right) = 0.$$

Next, for example, for a specific value of  $x = \frac{a_0 + a_1}{2}$ , membership function (7) takes the following form:

$$\mu_2(u) = \begin{cases} 0, u < 0, \\ \frac{(a_1 - a_0) - 2u(c - a_0)}{(1 - u)(a_1 - a_0)}, \\ 0 \leq u \leq \frac{a_1 - a_0}{2(c - a_0)}, \\ 0, u > \frac{a_1 - a_0}{2(c - a_0)}. \end{cases}$$

The chart of this membership function is shown in Fig. 4.

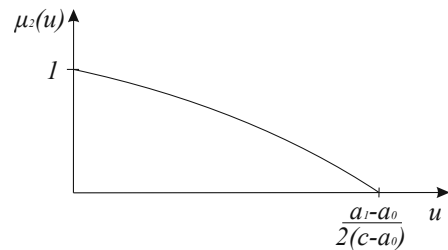


Fig. 4. Chart  $\mu_2(u)$  for  $x = \frac{a_1 + a_0}{2}$

Consider now the case where the membership function of the fuzzy number  $x$  is set in an infinite interval and takes, for example, the following form:

$$\mu(x) = \frac{1}{1 + \left(\frac{x - a}{\sigma}\right)^2}.$$

Assume that parameter  $a$  is a fuzzy number with membership function (2). As above, introduce:

$$u = \frac{1}{1 + \left(\frac{x-a}{\sigma}\right)^2}. \tag{8}$$

Find the membership function to a bifuzzy quantity  $u$ . Solving equation (8) relative to  $a$  produces the following:

$$a = x \pm \sigma \left(\frac{1-u}{u}\right)^{0.5}.$$

Assume  $x \leq a$ . Then  $a = x + \sigma \left(\frac{1-u}{u}\right)^{0.5}$ .

In this case, considering (2),

$$\mu_1(u) = \begin{cases} 0, x + \sigma \left(\frac{1-u}{u}\right)^{0.5} < a_0, \\ \frac{x + \sigma \left(\frac{1-u}{u}\right)^{0.5} - a_0}{a_1 - a_0}, \\ a_0 \leq x + \sigma \left(\frac{1-u}{u}\right)^{0.5} \leq a_1, \\ 0, x + \sigma \left(\frac{1-u}{u}\right)^{0.5} > a_1. \end{cases} \tag{9}$$

Next, the inequalities below follow from  $x + \sigma \left(\frac{1-u}{u}\right)^{0.5} < a_0$ :

$$\left(\frac{1-u}{u}\right)^{0.5} < \frac{a_0 - x}{\sigma};$$

$$\frac{1-u}{u} < \left(\frac{a_0 - x}{\sigma}\right)^2;$$

$$\frac{1}{u} < \left(\frac{a_0 - x}{\sigma}\right)^2 + 1;$$

$$u > \frac{1}{1 + \left(\frac{a_0 - x}{\sigma}\right)^2}.$$

By analogy, it is easy to show that:

– from  $x + \sigma \left(\frac{1-u}{u}\right)^{0.5} \geq a_0$  it follows  $u \leq \frac{1}{1 + \left(\frac{a_0 - x}{\sigma}\right)^2}$ ;

– from  $x + \sigma \left(\frac{1-u}{u}\right)^{0.5} \leq a_1$  it follows  $u \geq \frac{1}{1 + \left(\frac{a_1 - x}{\sigma}\right)^2}$ ;

– from  $x + \sigma \left(\frac{1-u}{u}\right)^{0.5} > a_1$  it follows  $u < \frac{1}{1 + \left(\frac{a_1 - x}{\sigma}\right)^2}$ .

Based on the ratios derived, we write (9) as follows:

$$\mu_1(u) = \begin{cases} 0, u < \frac{1}{1 + \left(\frac{a_1 - x}{\sigma}\right)^2}, \\ \frac{x + \sigma \left(\frac{1-u}{u}\right)^{0.5} - a_0}{a_1 - a_0}, \\ \frac{1}{1 + \left(\frac{a_1 - x}{\sigma}\right)^2} \leq u \leq \frac{1}{1 + \left(\frac{a_0 - x}{\sigma}\right)^2}, \\ 0, u > \frac{1}{1 + \left(\frac{a_0 - x}{\sigma}\right)^2}. \end{cases} \tag{10}$$

Calculate the value of  $\mu(u)$  at the edges of the interval:

$$\left[ \frac{1}{1 + \left(\frac{a_1 - x}{\sigma}\right)^2}, \frac{1}{1 + \left(\frac{a_0 - x}{\sigma}\right)^2} \right].$$

We obtain:

$$\begin{aligned} \mu_1\left(\frac{1}{1 + \left(\frac{a_1 - x}{\sigma}\right)^2}\right) &= \\ &= \left[ x + \sigma \left( \frac{\frac{1}{1 + \left(\frac{a_1 - x}{\sigma}\right)^2}}{\frac{1}{1 + \left(\frac{a_1 - x}{\sigma}\right)^2}} \right)^{0.5} - a_0 \right] \frac{1}{a_1 - a_0} = 1, \end{aligned}$$

$$\begin{aligned} \mu_1\left(\frac{1}{1 + \left(\frac{a_0 - x}{\sigma}\right)^2}\right) &= \\ &= \left[ x + \sigma \left( \frac{\frac{1}{1 + \left(\frac{a_0 - x}{\sigma}\right)^2}}{\frac{1}{1 + \left(\frac{a_0 - x}{\sigma}\right)^2}} \right)^{0.5} - a_0 \right] \frac{1}{a_1 - a_0} = 0. \end{aligned}$$

Find the  $\mu(u)$ 's membership function for specific values of  $x$ . Let  $x = a_0$ . Then:

$$\mu_1(u) = \begin{cases} 0, u < \frac{1}{1 + \left(\frac{a_1 - a_0}{\sigma}\right)^2}, \\ \frac{\sigma \left(\frac{1-u}{u}\right)^{0.5}}{a_1 - a_0}, \\ \frac{1}{1 + \left(\frac{a_1 - a_0}{\sigma}\right)^2} \leq u \leq 1. \end{cases}$$

The chart of the derived membership function is shown in Fig. 5.

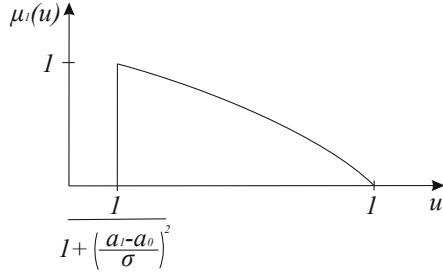


Fig. 5. Chart  $\mu_1(u)$  at  $x=a_0$

Now assume  $x=a$ . All required calculations are carried out in the same way as above, so we give only the final results:

$$\mu_2(u) = \begin{cases} 0, u < \frac{1}{1 + \left(\frac{x-a_0}{\sigma}\right)^2}, \\ \frac{x + \sigma \left(\frac{1-u}{u}\right)^{0.5} - a_0}{a_1 - a_0}, \\ \frac{1}{1 + \left(\frac{x-a_0}{\sigma}\right)^2} \leq u \leq 1. \end{cases}$$

The chart for the function  $x=a_1$  takes the form shown in Fig. 6.

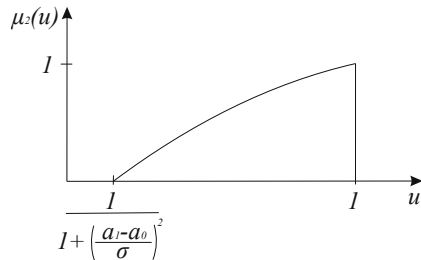


Fig. 6.  $\mu_2(u)$  membership function

Our examples illustrate the technique for deriving the membership functions to a bifuzzy quantity. In cases where not one but several fuzzy-quantity membership function parameters are also fuzzy with known membership functions, the resulting membership function can be obtained through the sequential application of the procedures described above.

### 6. An example of using the devised technique

Consider an example that clearly shows that if the original problem data are described in terms of fuzzy second-order numbers, accounting for this information is palpably useful.

Let the fuzzy demand  $x$  for some product is assessed by the following membership function:

$$\mu(x) = \begin{cases} 0, x < 0, \\ \frac{m_0 - x}{m_0}, 0 \leq x \leq m_0, \\ 0, x > m_0. \end{cases} \quad (11)$$

Consider an event where the  $m_0$  parameter of this membership function  $\mu(x)$  is a fuzzy number  $m$  with the following membership function:

$$\mu_{m_0}(m) = \begin{cases} 0, m < a, \\ \frac{m-a}{m_0-a}, a \leq m \leq m_0, \\ \frac{b-m}{b-m_0}, m_0 \leq m \leq b, \\ 0, m > b. \end{cases} \quad (12)$$

It is clear that each  $m$  fuzzy value corresponds to its own membership function to the fuzzy number  $x$ , the totality of which determines the membership functions to  $m \in [a, b]$ . We apply the proposed technique for calculating the membership functions to a bifuzzy number to the specific task of managing stocks. Introduce  $u = \frac{m-x}{m}$  and express the value of  $m$  through  $u$ :

$$m = \frac{x}{1-u}. \quad (13)$$

Next, we find the membership function to the fuzzy number  $u$ , depending on  $x$ , by substituting (13) in (12). We obtain:

$$\mu_{m_0}(u) = \begin{cases} 0, \frac{x}{1-u} < a, \\ \frac{\frac{x}{1-u} - a}{m_0 - a}, a \leq \frac{x}{1-u} \leq m_0, \\ \frac{b - \frac{x}{1-u}}{b - m_0}, m_0 \leq \frac{x}{1-u} \leq b, \\ 0, \frac{x}{1-u} > b. \end{cases} \quad (14)$$

The intervals of monotony of the variable  $u$  are recorded in a more familiar form. At the same time, if:

$$\frac{x}{1-u} < a,$$

then

$$x < a - au, \quad au < a - x, \quad u < \frac{a-x}{a};$$

if

$$\frac{x}{1-u} \geq a,$$

then

$$x \geq a - au, \quad au > a - x, \quad u > \frac{a-x}{a};$$

if

$$\frac{x}{1-u} < m_0,$$

then

$$x < m_0 - m_0u, \quad m_0u < m_0 - x, \quad u < \frac{m_0 - x}{m_0};$$

if

$$\frac{x}{1-u} \leq b,$$

then

$$x \leq b - bu, \quad bu \leq b - x, \quad u \leq \frac{b-x}{b};$$

if

$$\frac{x}{1-u} > b,$$

then

$$x > b - bu, \quad bu > b - x, \quad u > \frac{b-x}{b}.$$

Then

$$\mu_{m_0}(u) = \begin{cases} 0, & u < \frac{a-x}{a}, \\ \frac{\frac{x}{1-u} - a}{m_0 - a}, & \frac{a-x}{a} \leq u \leq \frac{m_0 - x}{m_0}, \\ \frac{b - \frac{x}{1-u}}{b - m_0}, & \frac{m_0 - x}{m_0} \leq u \leq \frac{b-x}{b}, \\ 0, & u > \frac{b-x}{b}. \end{cases} \quad (15)$$

Check if the  $\mu(u)$  description is correct.

If

$$\frac{\frac{x}{1-u} - a}{m_0 - a}, \quad u = \frac{a-x}{a},$$

then

$$\frac{\frac{x}{1 - \frac{a-x}{a}} - a}{m_0 - a} = \frac{\frac{xa}{m_0 - a} - a}{m_0 - a} = 0,$$

if

$$\frac{\frac{x}{1-u} - a}{m_0 - a}, \quad u = \frac{m_0 - x}{m_0},$$

then

$$\frac{\frac{x}{1 - \frac{m_0 - x}{m_0}} - a}{m_0 - a} = \frac{\frac{xm_0}{m_0 - a} - a}{m_0 - a} = 1,$$

if

$$\frac{b - \frac{x}{1-u}}{b - m_0}, \quad u = \frac{m_0 - x}{m_0},$$

then

$$\frac{b - \frac{x}{1 - \frac{m_0 - x}{m_0}}}{b - m_0} = \frac{b - \frac{xm_0}{m_0 - a}}{b - m_0} = 1,$$

if

$$\frac{b - \frac{x}{1-u}}{b - m_0}, \quad u = \frac{b-x}{b},$$

then

$$\frac{b - \frac{x}{1 - \frac{b-x}{b}}}{b - m_0} = \frac{b - \frac{bx}{b}}{b - m_0} = 0.$$

We shall use the resulting description (15) of the fuzzy number  $u$ 's membership function to estimate possible losses associated with an inaccurate description of the right limit of the possible demand values. To this end, fix the value of demand  $x_0$  within the bifuzziness interval  $[m_0, b]$ . Next, find the intervals of the  $\mu(u)$  membership function values that correspond to the possible values of the membership function for the chosen fuzzy  $x_0$ , and build a chart of the function  $\mu(u)$  for different demand values from the interval  $[m_0, b]$ .

Set specific parameters of the problem and analyze the behavior of the fuzzy number  $u$ 's membership function for different  $x$  values in the right-hand bifuzziness interval.

Variant A:  $a=10, m_0=20, b=40, x=22$ ,

Variant B:  $a=10, m_0=20, b=40, x=26$ ,

Variant C:  $a=10, m_0=20, b=40, x=30$ .

The corresponding charts of  $\mu_{m_0}(u)$  values are given in Fig. 7–9.

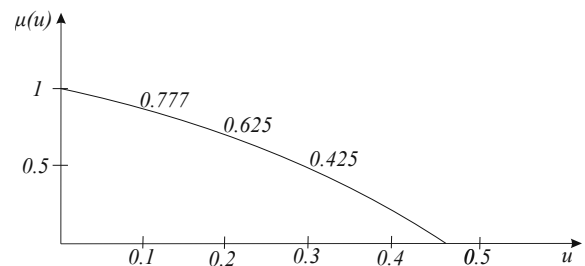


Fig. 7. Chart  $\mu(u)$ . Variant A

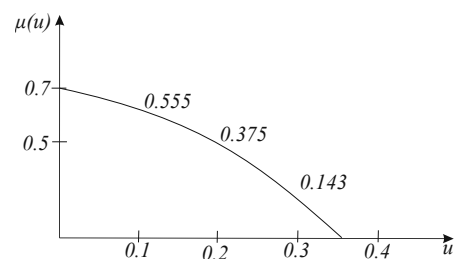


Fig. 8. Chart  $\mu(u)$ . Variant B

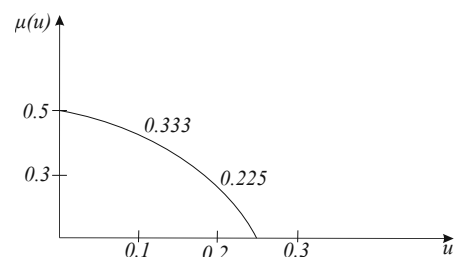


Fig. 9. Chart  $\mu(u)$ . Variant C

Our analysis of the above charts shows that the significant excesses of a fuzzy maximum value of demand correspond

to the significant values of the membership function of an actual bifuzzy demand. At the same time, it is clear that making a decision, based on the description of demand in form (11), can lead, in this case, to significant losses.

Thus, the proposed method of calculating the membership function for bifuzzy numbers is an important way to expand the arsenal of fuzzy mathematics. This method provides an opportunity to build adequate models of systems and processes that take into consideration the actual hierarchy of the uncertainty in the original data.

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**6. Discussion of results of devising a method for calculating the membership functions of bifuzzy numbers**

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The result of our study is a general approach proposed to obtain an analytical description of the membership functions to bifuzzy numbers. The fundamental advantage of the proposed method for constructing bifuzzy numbers' membership functions is to obtain an analytical description of these functions in the form that is adapted and convenient for practical use in solving specific problems. The resulting ratio:

$$\mu_1(u) = \begin{cases} 0, u < \frac{x-a_1}{c-a_1}, \\ \frac{c-a_0 + \frac{c-x}{u-1}}{a_1-a_0}, \\ \frac{x-a_1}{c-a_1} \leq u \leq \frac{x-a_0}{c-a_0}, \\ 0, u > \frac{x-a_0}{c-a_0} \end{cases}$$

determines this membership function for a specific case where one of the initial fuzzy number's parameters is set vaguely over the finite interval.

Applying the proposed basic technique for fuzzy numbers with an infinite carrier has allowed us to derive a ratio for the membership function to a specific bifuzzy number. The analytical description of this function is brought to a standard form.

$$\mu_1(u) = \begin{cases} 0, x + \sigma \left( \frac{1-u}{u} \right)^{0.5} < a_0, \\ \frac{x + \sigma \left( \frac{1-u}{u} \right)^{0.5} - a_0}{a_1 - a_0}, \\ a_0 \leq x + \sigma \left( \frac{1-u}{u} \right)^{0.5} \leq a_1, \\ 0, x + \sigma \left( \frac{1-u}{u} \right)^{0.5} > a_1. \end{cases}$$

The recent increase in the number of publications on the issue related to solving problems under the conditions of uncertainty is objectively motivated. One important reason is a clear understanding of the lack of validity for the rigid need to use traditional methods and techniques from a probability theory in many practical tasks in the face of uncertainty of the raw data. A characteristic feature of the tasks of studying phenomena and objects in the actual world is the objective heterogeneity of these data. Given the variability of the most

important defining properties and characteristics of the medium, practical tasks are stated and solved in small samples when fundamental provisions are violated, and the axiomatic basis of the probability theory. Changing the mechanism that forms the phenomenon of randomness leads to heterogeneity of samples containing a large number of elements, not sufficiently representative of the general population. The alternative related to the works by L. Zadeh is the theory of fuzzy sets [1], which is much less demanding of the quality of the raw data; it makes it possible to solve practical problems and draw reasonable conclusions for actual samples. At the same time, to describe the totalities of fuzzy numbers, membership functions are used – informative analogs of probability distributions of random quantities.

In accordance with that, this work has proposed an authentic technique for calculating analytical descriptions of the membership functions of fuzzy numbers, the parameters of which are also described vaguely. In the implementation of this technique, the bifuzzy numbers from the alphabetical object of the general theory of fuzzy sets are transformed into a working tool for practical use. It has been proven that a bifuzzy number introduced in this way corresponds not to one but to the family of membership functions. Using a specific inventory management task as an example, it has been shown that correct accounting of the actual bifuzziness of demand avoids possible tangible losses.

Examples of the application of the devised method for fuzzy numbers with the finite and infinite carrier are given. They demonstrate that the complexity of analytic description of membership functions with hierarchical uncertainty is growing rapidly with the increasing number of parameters for the membership function of an original fuzzy number, which are also set vaguely.

A possible area of further research is to advance techniques for solving optimization tasks under the conditions of bifuzzy uncertainty. In that case, the approach proposed in [22] could prove useful.

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**7. Conclusions**

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1. Using an analytical description of fuzzy numbers for cases where one of the parameters of these numbers is set vaguely, a basic technique has been proposed to describe the emerging bifuzzy number. Based on this technique, we have proposed and substantiated an analytical method of constructing the membership functions of bifuzzy quantities. The proposed technique fills the gap in fuzzy sets theory. The method makes it possible, based on the original hierarchical description of a bifuzzy number, to derive an analytical expression for its membership function in a standard form. This description of the membership functions allows such numbers to be used constructively when solving specific tasks. The practical significance of our result is the prospect of its application to improve the adequacy of models of systems operating in uncertain settings.

2. The proposed method has been implemented relative to bifuzzy numbers with the finite and infinite carrier. The computational scheme that implements the method is easily generalized for the case when the number of fuzzy parameters of the original number exceeds one. The basic devised technique to construct membership functions of the second-order fuzzy numbers can be used to solve the same problem relative to fuzzy numbers, the order of which is higher than the second order.



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