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FUEL-OPTIMAL CONTROL OF SYSTEM WITH DELAY

The paper deals with the synthesis of the fuel-optimal control for the systems with delay in control loop. The considerations are confined to the class of the linear time-invariant systems performed by (1, 2). Synthesis of the optimal control is achieved using the Pontryagin's maximum principle.

I. Introduction

Ever increasing demands for the growth of production rate, and for improved quality of the products have imposed an intense mechanization of production.

The increase in production rate necessitates the realization of high speed-programmed control of the variables of the process considered. In such cases the time between the measurement of their values and the control realization (delay in control loop) becomes one of the most important factors.

Time-delay appears to be a dominant factor in systems of high speed aireraf ts, rockets, and in other large-scale systems, e.g. in some processes.

An urgent task for such systems is the synthesis of fuel-optimal control indispensable either due to limited resources, or to minimization of fuel consumption.

II. Statement of the problem

We consider the class of the linear time-invariant systems, which may be presented by a consistent combination of the first-order linear dynamic links:

$$\frac{dx(t)}{dt} = Ax(t) + bu(t - \Theta); t \in [t_0, T] \qquad \text{(II-1)}$$

and

$$\frac{dx(t)}{dt} = A_1x(t) + A_2x(t - \tau) + bu(t); t \in [t_0, T] \qquad \text{(II-2)}$$

where: $x(t) = (x1(t), x2(t))$ – is a state vector; $u(t)$ – is a scalar control, which is subjected to restriction.

$$|u(t)| < 1; \qquad \text{(II-3)}$$

$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ – is a column vector; A, $A1$, $A2$ — are 2x2 constant matrices; τ, $\Theta =$ *constant* > 0.

The control problem consists in seeking a scalar control $u(t)$ and trajectory $x(t)$ satisfying the restriction (II-3) and initial functions, respectively:

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$$x(t) = \mu(t);\ t \in [t_0 - \tau, t_0],$$
$$u(t) = \varphi(t);\ t \in [t_0 - \Theta, t_0] \qquad \text{(II-4)}$$

and to transfer the systems (II-l) and (II-2) from an initial state $x(0) = (x_{10}, x_{20})$ at the time interval $t \in [t0, T]$ to a terminal state $x(T) = (0, 0)$ by minimizing the functional:

$$\text{F(u(t))} = \int_{t_0}^{T} [K + |u(t)|]\, dt \qquad \text{(II-5)}$$

in the following cases:

1. $K = \text{const}, T$ — unfixed terminal time;
2. $K = 0, T$ — fixed terminal time.

III. Synthesis of optimal control

The solution of the problem is based on the maximum principle of Pontryagin. From application of this principle it follows that there exist two dimensional not identically zero and continuous vector-function $\psi(t)$, and an optimal control $u(t)$, which minimize Hamilton function:

$$H = K + |u(t)| + \langle \psi, x \rangle \qquad \text{(III-1)}$$

Function $\psi(t)$ satisfies moreover differential equations;

$$\frac{d\psi_i(t)}{dt} = -\frac{\partial H}{\partial x_i} - \left[\frac{\partial H}{\partial x_i(t-\tau)}\right]^{t+\tau};\ (t_0 \leqslant t \leqslant T - \tau) \qquad \text{(III-2)}$$

$$\frac{d\psi_i(t)}{dt} = -\frac{\partial H}{\partial x_i};\ (T - \tau \leqslant t \leqslant T); (i = 1, 2) \qquad \text{(III-3)}$$

and the initial condition $\psi(t_0) = \psi_0$.

If the function $\psi(t)$ is known then the optimal control is determined due the minimizing Hamiltonian H, and can be written as:

$$u*(t) = 0,\ \ if|q(t)| < 1;$$
$$u*(t) = -sign\{q(t)\},\ \ if|q(t)| \quad > 1, \qquad \text{(III-4)}$$

where q(t) =$\psi_2(t)$.

For comparative reasons let us consider — without any loss of generality the fuel-optimal problem for all cases of performance criterion (II-5), and for the plants, which can be represented by two integrators with delay in control loop or state.

IV. Double integral plants with delay in control loop

These plants can be described by the vector differential equation (II-l), where the matrix A is given as:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Case 1. The index of performance (II-5) is the following:

$$F(u(t)) = \int_{t_0}^{T} [K + |u(t)|]\, dt \tag{IV-1}$$

where $K > 0$ and T — is unfixed time.

It is first necessary to consider a similar problem for the plant described by the equation (II-l) assuming that $\Theta = 0$. Using the approach [1] we can obtain an analytical form of switching curves, which divide the phase plane $(x1, x2)$ into some regions, where the optimal control is simple and equal ± 1 or to zero.

If the algorythmes are applied to plant (II-l) then — because of time-delay in a control loop — $u(t)$ does not change its value on the switching curves earlier than after expiration time Θ. This leads to limited cycle performance in the system.

According to [2] the time-delay can be compensated by a predictive change of time-delay control $u(t)$. Thus the switching curves of the system (II-l) should be established as a geometrical place of points which may be achieved during the time Θ by reflection of corresponding switching curve obtained for $\Theta = 0$.

On basis of the method, discussed above, for given system the following switching curves have been obtained:

$$\begin{aligned} \Gamma^+ &= \left\{(x_1, x_2) : x_1 = \tfrac{K+4}{2K} x_2^2 - x_2\Theta \left(1 + \tfrac{K+4}{K}\right) + \tfrac{\Theta^2}{2}\left(1 + \tfrac{K+4}{K}\right)\right\} \\ \Gamma^- &= \left\{(x_1, x_2) : x_1 = -\tfrac{K+4}{2K} x_2^2 - x_2\Theta \left(1 + \tfrac{K+4}{K}\right) - \tfrac{\Theta^2}{2}\left(1 + \tfrac{K+4}{K}\right)\right\} \end{aligned} \tag{IV-2}$$

$$\gamma^+ = \left\{(x_1, x_2) : x_1 = \frac{(x_2 - \Theta)^2 - \Theta^2}{2}\right\}; \Delta t \geqslant \Theta \tag{IV-3}$$

$$\gamma^- = \left\{(x_1, x_2) : x_1 = -\frac{(x_2 - \Theta)^2 - \Theta^2}{2}\right\};$$

$$\gamma^+ = \left\{(x_1, x_2) : x_1 = \frac{1}{2} x_2^2 \frac{K-2}{K+2} - x_2\Theta\frac{2K}{K+2} + \frac{K}{K+2}\Theta^2\right\}; \Delta t \leqslant \Theta$$

$$\gamma^- = \left\{(x_1, x_2) : x_1 = -\frac{1}{2} x_2^2 \frac{K-2}{K+2} - x_2\Theta\frac{2K}{K+2} + \frac{K}{K+2}\Theta^2\right\}.$$

On these curves the control signal $u(t)$ changes its magnitude according to the rule:

1. from $u = +1 (u = -1)$ to $u = 0$ on the line $\Gamma^+(\Gamma^-)$

2. from $u = 0$ to $u = +1 (u = -1)$, on the line $\gamma^+(\gamma^-)$. Δt denotes the time movement on plant's trajectories from curves Γ to curves γ and the conditions for Δt, given above, are necessary.

The condition of equilibrium state for the system is the following:

$$u^*(t) = 0; \quad (T - \Theta \leqslant t \leqslant T) \tag{IV-4}$$

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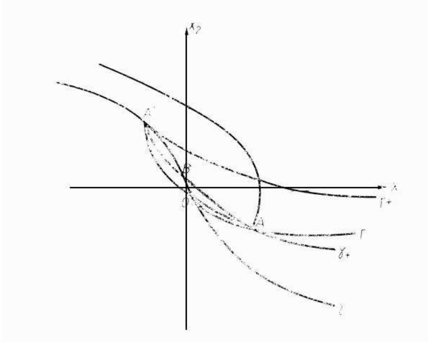

Fig. 1. – Phase portrait double integral plants with delay in control loop.

Case 2. The index of performance (II-5) is the following:

$$F(u(t)) = \int_{t_0}^{T_f} u(t)dt,$$

where Tf is fixed time.

For simplicity of systems trajectory determing at time interval $0 < t < \Theta$ let us assume that $\varphi(t)$ is a cost function and that its value coincides with the magnitude $u*(t)$ till the first moment of switching. In this case compensation of time delay (similarly to the case 1) can be realized by applying (during the time-delay) the predicted magnitude of optimal control $u*(t)$ which was obtained in [1] for system without time-delay.

Finally the condition (IV -4) will be satisfied, if the terminal switching in a system without delay does not occur at time interval:

$$T_f - \Theta \leqslant t \leqslant T_f.$$

V. Double integral plants with state delay

These plants are described by the system (II-2) where the matrices A1 and A2 are:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

From Hamiltonian:

$$H = K + |u(t)| + \psi_1(t)x_1(t-\tau) + \psi_2(t)u(t) \tag{V-1}$$

we get adjoint system:

$$\begin{array}{ll} \dot{\psi}_1(t) = 0; & (t_0 \leqslant t \leqslant T); \\ \dot{\psi}_2(t) = -\psi_1(t+\tau); & (t_0 \leqslant t \leqslant T-\tau); \\ \dot{\psi}_2(t) = 0; & (T-\tau \leqslant t \leqslant T) \end{array} \tag{V-2}$$

with the following decisions:

$$\begin{aligned} \psi_1(t) &= \psi_{10}; \quad (t_0 \leqslant t \leqslant T); \\ \psi_2(t) &= -\psi_{10}t + \psi_{20}; \quad (t_0 \leqslant t \leqslant T-\tau); \\ \psi_2(t) &= -\psi_{10}T + \psi_{20}; \quad (T-\tau \leqslant t \leqslant T); \end{aligned} \tag{V-3}$$

In accordance with (III-3), (V-l) the optimal control is:

$$u^*(t) = -sign\left\{\psi_2(t)\right\},\ \text{if}|\psi_2(t)| > 1; \tag{V-4}$$

$$u^*(t) = 0, \text{if}|\psi_2(t)| < 1;$$

The condition of equilibrium state of the system:

$$x2(t) = 0,\ (T \geqslant t \geqslant T-\tau)\text{and}u*(t) = 0$$

Case 1. The index performance (II-5) is the following:

$$F(u(t)) = \int_{t_0}^{T} \left[K + |u(t)|\right] dt$$

where $K > 0$, and T is unfixed time.

To find the optimal control curves we use the method of "reverse time". Let us introduce "reverse time"defined as $z = T - t$. In this case the system can be written in the following form:

$$\begin{aligned} \dot{x}_1(z) &= -x_2(z+\tau); \\ \dot{x}_2(z) &= -u(z); \end{aligned} \tag{V-5}$$

and

$$\begin{aligned} x_1(z) &= 0; \quad (0 \leqslant z \leqslant \tau); \\ x_2(z) &= \varphi(T-z); \quad (T \leqslant z \leqslant T+\tau); \end{aligned} \tag{V-7}$$

From (V-3), (V-4) we conclude, that in an ordinary case the optimal sequence of controls is $u0, 0, -u0(u0 = \pm 1)$, where the magnitude changes in the moments of time $z1$ and $z2$.

Then in view of (V-6) and (V-7) the movement of the system in the phase plane starting from the origin is as follows:

$$x_2(z) = -u_0 z + u_0\tau; \quad (\tau \leqslant z \leqslant z_1); \tag{V-8}$$

$$x_2(z) = -u_0 z_1 + u_0\tau; \quad (z_1 \leqslant z \leqslant z_2); \tag{V-9}$$

$$x_2(z) = +u_0 z - u_0(z_1 + z_2 - \tau); (z_2 \leqslant z \leqslant T) \tag{V-10}$$

$$x_1 = \frac{u_0 z^2}{2}; \quad (0 \leqslant z \leqslant z_1 - \tau) \tag{V-11}$$

$$x_1 = u_0(z_1-\tau) - \frac{u_0(z_1-\tau)^2}{2}; \quad (z_1 - \tau \leqslant z \leqslant z_2 - \tau) \tag{V-12}$$

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$$x_1 = -\frac{u_0(z+\tau)^2}{2} + u_0(z_1 + z_2 - \tau) - \frac{u_0(z_1-\tau)^2}{2} - \frac{u_0(z_2-\tau)^2}{2} + \frac{u_0\tau^2}{2};$$

$$(z_2 - \tau \leqslant z \leqslant T - \tau) \qquad \text{(V-13)}$$

It is clear from figure 2 that two cases of control are possible, when ***z2* —** ***z1***$> \tau$ and ***z2* —*z1*** $< \tau$.

Solving jointly the relations (V-8) and (V-12) for the moment of time $z = z1$ we find the optimal switching curve γ_1 (fig. 2):

$$x_1 = \frac{1}{2u_0}x_1^2 - x_2\tau; \qquad \text{(V-14)}$$

using the condition that $z_2 - z_1 \geqslant \tau$, where control reverse from zero into $u = u_0$.

Equating the value of Hamiltonian (V-1) to zero along the optimal path and using

$$\psi_2(z_2) - \psi_2(z_1) = \psi_1(z_2 - z_1) = 2u_0$$

we find that:

$$z_2 - z_1 = -\frac{2x_2u_0}{K} \qquad \text{(V-15)}$$

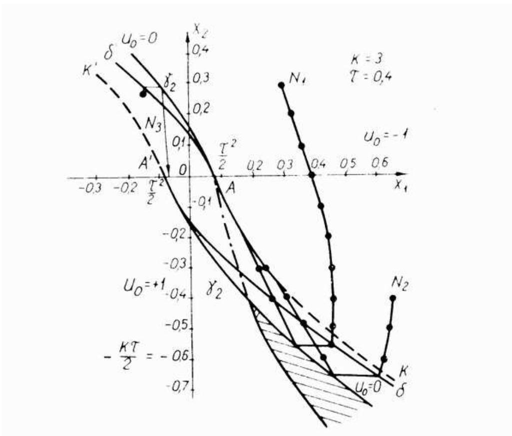

Fig. 2. – Phase portrait double integral plants with state delay.

Then solving jointly the relations (V-8), (V-13) and (V-15) for the moment of time $z = z1$ we find optimal switching curve γ_2 (fig. 2).

$$x_1 = \frac{K^2 - 4u_0^2}{2u_0K^2}x_2^2 - x_2\tau\left(1 + \frac{2}{K}\right) - \frac{u_0\tau^2}{2}; \qquad \text{(V-16)}$$

where control reverse from zero into $u = u_0$.

The relation (V-16) holds when $z2 - z1 \leqslant \tau$ or — comparing (V-14) and for:

$$x_2 \leqslant -\frac{K\tau}{2u_0}$$

According to figure 2 for the moment time $z = z2$ (when the control is switched from — $u0$ to zero) and for the opposite moment of time $z = z1$ there exists the unique optimal switching curve δ (fig. 2):

$$x_1 = \frac{K + 4u_0^2}{2u_0}x_2^2 - x_2\tau - \frac{u_0\tau^2}{2} \quad \text{(V-17)}$$

Which can be determined by resolving of (V-10), (V-13), (V-15) jointly.

Case 2. The index of performance (II-5) is the following:

$$F(u(t)) = \int_{t_0}^{T_f} |u(t)|dt$$

where T_f is fixed.

Let us assume $\mu(t) = \text{const} = x_{20}$ where $x_{20} = x_2(0)$. Moreover from the demands with respect to the terminal state of the system we have

$$u*(t) = x_2(t) = 0, \quad (T_f - \tau \leqslant t \leqslant T_f) \quad \text{(V-18)}$$

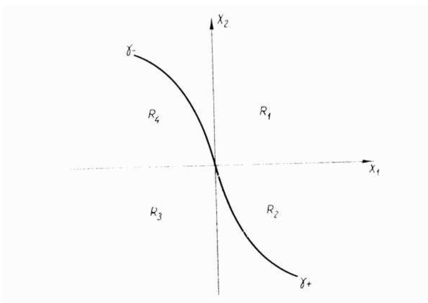

Fig. 3. – Phase portrait of the optimum control system with the fixed time of transition.

The solution of original system equations with the boundary conditions performed jointly with (V-18), and with T_f fixed yields inequalities:

$$T_f \geqslant \frac{x_{20}}{u_0} + \tau + \left[\frac{4}{u_0}(x_{10} + x_{20}\tau) + \frac{2}{u_0^2}x_{20}^2\right]^{1/2}$$

for $(x_{10}, x_{20}) \in R_1 \cup R_2 \cup R_3 \cup R_4$;

$$T_f \geqslant \left|\frac{x_{20}}{u_0}\right|, \text{for}(x_{10}, x_{20}) \in \gamma_+ \cup \gamma_-;$$

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Where:

$$\gamma_+ = \gamma_- = \left\{(x_1, x_2) : x_1 = \frac{1}{2u_0}x_2^2 - x_2\tau\right\}$$

$$u_0 = +1; \mathbf{for}(x_{10}, x_{20}) \in R_1 \cup R_4 \cup \gamma_+$$

and

$$u_0 = -1, \mathbf{for}(x_{10}, x_{20}) \in R_2 \cup R_3 \cup \gamma_-$$

Now we can determine the optimal control in the following way:

1. For all $(x_{10}, x_{20}) \in R_1 \cup R_3$ and for any Tf or for all:

$$(x_{10}, x_{20}) \in R_2 \cup R_4 \mathbf{and} T_f \leqslant T* = -\frac{1}{2u_0}x_{10} - \frac{x_{10}}{x_{20}}$$

the optimal control is unique and equal to:

$$u*(t) = \begin{cases} -u_0, & \mathbf{for}\, 0 \leqslant t \leqslant t_1 \\ 0, & \mathbf{for}\, t_1 \leqslant t \leqslant t_2 l \\ +u_0, & \mathbf{for}\, t_2 \leqslant t \leqslant T_f - \tau \end{cases}$$

where:

$$t_1 = \frac{1}{2}\left\{T_f + \frac{x_{20}}{u_0} - \tau - \left[\left(T_f - \frac{x_{20}}{u_0} - \tau\right)^2 - \frac{u}{u_0}(x_{10} + x_{20}\tau) - \frac{2}{u_0^2}x_{20}^2\right]^{1/2}\right\},$$

$$t_2 = \frac{1}{2}\left\{T_f + \frac{x_{20}}{u_0} - \tau + \left[\left(T_f - \frac{x_{20}}{u_0} - \tau\right)^2 - \frac{u}{u_0}(x_{10} + x_{20}\tau) - \frac{2}{u_0^2}x_{20}^2\right]^{1/2}\right\},$$

2. For all $(x10, x20) \quad \in R_2 \cup R_4$ and for any:

$$T_f > T* = -\frac{1}{2u_0}x_{20} - \frac{x_{10}}{x_{20}}$$

the optimal control is not unique.

3. For all $(x10, x20) \quad \in \gamma_+ \cup \gamma_-$ the optimal control is unique and equal to:

$$u*(t) = \begin{cases} -\mathrm{sign}\left\{x_{20}\right\}, & \mathbf{for}\, 0 \leqslant t \leqslant \left|\frac{x_{20}}{u_0}\right|; 1\left|\frac{x_{20}}{u_0}\right| < \tau; \\ 0, & \mathbf{for}\, \left|\frac{x_{20}}{u_0}\right| \leqslant t \leqslant T_f. \end{cases}$$

Remark, that the law obtained is governed by the conditions $t_1 > \tau$ and $t_2 \leqslant T_f - 2\tau$.

VI. Conclusion

Case 1.

1. The optimal control is bang-bang with one or two switchings.
2. The optimal control curves have the detection property.

3. The presence of time-delay Θ or τ in the system leads to the regions with nonoptimal control (the system with fixed function $\mu(t)$ or $\varphi(t)$ and with initial state (x_{10}, x_{20}) from these regions cannot achieve the origin of phase plane; For this purpose it is necessary to realize the switching in the first Θ or τ seconds.
4. If the initial point is placed in the nonoptimal region, then the optimal control in the interval $(0 \leqslant t \leqslant \Theta)$, or, $(0 \leqslant t \leqslant \tau)$ depends on the initial function $\mu(t)$ or $\varphi(t)$ as well as on the place of point on the phase plane. Moreover in this case there a singular switching curve does not exist.

Case 2.

1. The optimal control is bang-bang with one or two switchings.
2. The optimal control is not only the function of the initial state and time response, but also the function of time-delay $\Theta(\,or\ \tau)$ and depends significantly on the given $\mu(t)\ \varphi\ (or\ (t))$.

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