doi: https://doi.org/10.15407/dopovidi2018.04.009
UDC 517.5

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On blow-up solutions and dead zones in semilinear equations

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We study semilinear elliptic equations of the form $\operatorname{div}(A(z)\nabla u) = f(u)$ *in* $\Omega \subset \mathbf{C}$, *where* $A(z)$ *stands for a symmetric* 2×2 *matrix function with measurable entries,* $\det A = 1$, *and such that* $1/K|\xi|^2 \leqslant \langle A(z)\xi, \xi\rangle \leqslant K|\xi|^2, \xi \in \mathbf{R}^2$, $1 \leqslant K < \infty$. *Making use of our Factorization theorem, we give some explicit solutions for the above equation if* $f = e^u$ *or* $f = u^q$, *when matrices* $A(z)$ *are chosen in an appropriate form.*

Keywords: *quasiconformal mappings, semilinear PDE, blow-up solutions.*

1. Introduction. In this paper, we give new applications of the quasiconformal mappings theory, see e.g. [1—6], to the study of semilinear partial differential equations in the plane.

Let Ω be a domain in the complex plane $\mathbf{C}$. Denote by $M^{2\times 2}(\Omega)$ the class of two by two symmetric matrices $A(z) = \{a_{jk}\}$ with measurable entries and $\det A(z) = 1$ almost everywhere in Ω satisfying the uniform ellipticity condition

$$\frac{1}{K}|\xi|^2 \leqslant \langle A(z)\xi, \xi\rangle \leqslant K|\xi|^2 \text{ a.e.} \quad \text{in} \quad \Omega, \tag{1}$$

for every $\xi \in \mathbf{C}$. The factor K can be either a constant $1 \leqslant K < \infty$ or a measurable function $K(z) \in L^\infty(\Omega)$, with $1 \leqslant K(z) < \infty$ a.e. in Ω. Every such matrix function A generates a quasiconformal mapping ω as a homeomorphic solution of the Sobolev class $W^{1,2}_{\text{loc}}(\Omega)$ to the Beltrami equation

$$\omega_{\bar z}(z) = \mu(z)\omega_z(z) \quad \text{a.e.} \quad \text{in} \quad \Omega, \tag{2}$$

where the complex dilatation $\mu(z)$ is given by

$$\mu(z) = \frac{1}{\det(I + A)}(a_{22} - a_{11} - 2ia_{12}). \tag{3}$$

The condition of ellipticity (1) is written now as

$$|\mu(z)| \leqslant \frac{K-1}{K+1} \quad \text{a.e.} \quad \text{in} \quad \Omega. \tag{4}$$

Vice versa, given a measurable complex-valued function μ satisfying (4), we can invert the algebraic system (3) to obtain

$$A(z)=\begin{pmatrix} \frac{|1-\mu|^2}{1-|\mu|^2} & \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} \\ \frac{-2\operatorname{Im}\mu}{1-|\mu|^2} & \frac{|1+\mu|^2}{1-|\mu|^2} \end{pmatrix}. \tag{5}$$

In this case, we say that the matrix function A and the corresponding quasiconformal mapping ω *are agreed.*

Let $f:\mathbf{R}\to\mathbf{R}$ be a continuous function. In [7] we have proven the following Factorization theorem, cf. the corresponding result for the smooth case in [8].

Theorem 1. *Let* $A(z)\in M^{2\times 2}(\Omega)$. *Then every weak solution* $u\in W^{1,2}_{\mathrm{loc}}(\Omega)\cap C(\Omega)$ *of the semilinear equation*

$$\operatorname{div}[A(z)\nabla u(z)]=f(u(z)),\quad z\in\Omega, \tag{6}$$

can be expressed as

$$u(z)=T(\omega(z)), \tag{7}$$

where $\omega:\Omega\to G$ *is a K-quasiconformal mapping agreed with the matrix function* $A(z)\in M^{2\times 2}(\Omega)$ *and* $T\in W^{1,2}_{\mathrm{loc}}(G)\cap C(G)$ *is a weak solution to the equation*

$$\Delta T=J(w)f(T(w)),\quad a.e.\quad in\quad G. \tag{8}$$

Here, $J(w)$ *stands for the Jacobian determinant of the inverse mapping* $z=\omega^{-1}(w)$.

Among the quasiconformal mappings $\omega:\Omega\to G$, there are a variety of the so-called volume-preserving maps, for which $J_\omega(z)\equiv 1$, $z\in\Omega$. In this partial case, we arrive at the following statement:

Corollary 1. *Let* $A(z)\in M^{2\times 2}(\Omega)$ *be a matrix function that generates a volume-preserving quasiconformal mapping* $\omega(z)$. *Then every weak solution* $u\in W^{1,2}_{\mathrm{loc}}(\Omega)\cap C(\Omega)$ *of the semilinear equation*

$$\operatorname{div}[A(z)\nabla u(z)]=f(u(z)),\quad z\in\Omega, \tag{9}$$

can be expressed as

$$u(z)=T(\omega(z)), \tag{10}$$

where $T\in W^{1,2}_{\mathrm{loc}}(G)\cap C(G)$ *is a weak solution to the equation*

$$\Delta T=f(T(w)),\ a.e.\ in\ G. \tag{11}$$

Some applications of the Factorization theorems that we are going to give below are based just on Corollary 1.

2. Explicit blow-up solutions. Let Ω be a bounded domain in $\mathbf{C}$ and let $\partial\Omega$ denote its boundary. In this section, we study the problem

$$\operatorname{div}[A(z)\nabla u(z)]=f(u(z)), \tag{12}$$

$$u(z)\to\infty,\quad \text{as}\quad d(z):=\operatorname{dist}(z,\partial\Omega)\to 0, \tag{13}$$

see, e.g., [9] and [10], as well as its Laplace's counterpart:

$$\nabla u(z) = f(u(z)), \tag{14}$$

$$u(z) \to \infty, \quad \text{as} \quad d(z) := \operatorname{dist}(z, \partial\Omega) \to 0. \tag{15}$$

Solutions to these problems are called *boundary blow-up solutions* or *large solutions*. If $f(u) = e^u$, then (14) is a classical Liouville—Bieberbach semilinear equation that was first investigated by Bieberbach in his pioneering work [11] related to the study of automorphic functions in the plane. The corresponding equation (12) with $f(u) = e^u$, can be viewed as a divergent counterpart to the Liouville—Bieberbach semilinear equation. Recall that if f is a conformal mapping of Ω onto the unit disk, then the boundary blow-up solutions for the Liouville—Bieberbach semilinear equation are expressed explicitly by the formula

$$u(z) = \log \frac{8|f'(z)|^2}{(1-|f(z)|^2)^2}. \tag{16}$$

Theorem 2. *Let Ω be the annulus $r < |z| < 1$ in the complex plane $\mathbf{C}$ and let the matrix function $A(z) \in M^{2\times 2}(\Omega)$ is generated by the formula* (5) *with the complex dilatation*

$$\mu(z) = \left(\nu^2(|z|) \pm \nu(|z|)\sqrt{1-\nu^2(|z|)}\right)\frac{z}{\bar{z}}, \tag{17}$$

where $\nu(t)$, $0 \leqslant t < 1$, stands for an arbitrary measurable function. If $|\nu(t)| \leqslant q < 1$, then there exists one and only one boundary blow-up solution to the semilinear equation

$$\operatorname{div}[A(z)\nabla u] = e^u \ \text{ in the annulus } r < |z| < 1, \tag{18}$$

which is given explicitly by the formula

$$u(z) = \log \frac{2\pi^2}{|z|^2 \log^2 r \cdot \sin^2\left(\dfrac{\pi}{\log r}\log|z|\right)}. \tag{19}$$

Indeed, if the complex dilatation $\mu(z)$ has the form

$$\mu(z) = k(|z|)\frac{z}{\bar{z}}, \tag{20}$$

where $k(\tau): \mathbf{R} \to \mathbf{C}$ is a measurable function such that $|k(\tau)| \leqslant k < 1$, then the formula

$$\omega(z) = \frac{z}{|z|}\exp\left\{\int_1^{|z|} \frac{1+k(\tau)}{1-k(\tau)}\frac{d\tau}{\tau}\right\} \tag{21}$$

represents a unique quasiconformal mapping of the unit disk, as well as the whole complex plane, onto itself with complex dilatation μ and the normalization: $\omega(0) = 0$, $\omega(1) = 1$, see, e.g., [4, p. 82], and [12].

Analyzing formula (21) with specified as above $k(t)$, we see that the Jacobian $J_\omega(z) \equiv 1$, i.e., the mapping ω is volume-preserving, and $|\omega(z)| = |z|$ for $z \in \mathbf{C}$. Mapping conformally the given annulus onto the unit disk and applying the Bieberbach explicit formula (16), we see that the function

$$T(w)=\log\frac{2\pi^2}{|w|^2\log^2 r\cdot\sin^2\left(\dfrac{\pi}{\log r}\log|w|\right)} \qquad (22)$$

represents the blow-up solution to the semilinear Liouville—Bieberbach equation in the annulus $r<|w|<1$. It remains to apply Corollary 1. The uniqueness follows from a fundamental result by Marcus and Véron, see [10], Theorem 5.3.7.

Our next example deals with the study of the blow-up solutions to the Liouville—Biberbach type equation defined in an unbounded domain of the complex plane.

Theorem 3. *Let* H^+ *be the right half-plane* $\{z:\operatorname{Re} z>0\}$ *in the complex plane* $\mathbf{C}$ *and let the matrix function* $A(z)\in M^{2\times 2}(\Omega)$ *have the entries* $a_{11}=1$, $a_{12}=\pm 2\nu(x)/\sqrt{1-\nu^2(x)}$, $a_{22}=(1+3\nu^2(x))/(1-\nu^2(x))$, *where* $\nu(x)$, $x\in\mathbf{C}$, *stands for an arbitrary measurable real-valued function such that* $|\nu(x)|\leqslant q<1$. *Then there exist boundary blow-up solutions to the semilinear equation*

$$\operatorname{div}[A(z)\nabla u]=e^u,\quad z\in H^+, \qquad (23)$$

which are written explicitly:

$$u(z)=\log\frac{2}{x^2},\quad z=x+iy, \qquad (24)$$

$$u(z)=\log 8\lambda^2-2\lambda x-2\log(1-e^{-2\lambda x}),\quad \lambda>0. \qquad (25)$$

Indeed, the matrix function $A(z)$ with the above entries generates, by formula (3) the complex dilatation

$$\mu(z)=(\nu^2(x)\pm i\nu(x)\sqrt{1-\nu^2(x)}) \qquad (26)$$

which, as we see, depends on x only. By Proposition 5.23 in [4], see also [12], a unique quasiconformal mapping of the right half-plane onto itself with the complex dilatation μ and the normalization $\omega(0)=0$, $\omega(i)=i$ and $\omega(\infty)=\infty$, is represented explicitly by the formula

$$\omega(z)=\int_0^x\frac{1+\mu(t)}{1-\mu(t)}dt+iy. \qquad (27)$$

Analyzing formula (27), we see that the Jacobian $J_\omega(z)\equiv 1$, i. e., the mapping ω is volume-preserving, and $\operatorname{Re}\omega(z)=x$ for $z\in\mathbf{C}$. By Corollary 1, a solution $u\in W^{1,2}_{\mathrm{loc}}(\Omega)\cap C(\Omega)$ of the semilinear equation

$$\operatorname{div}[A(z)\nabla u(z)]=e^u,\quad z\in H^+, \qquad (28)$$

is expressed as

$$u(z)=T(\omega(z)), \qquad (29)$$

where $T\in W^{1,2}_{\mathrm{loc}}(G)$ is a solution to the equation

$$\Delta T(w)=e^{T(w)},\quad in\quad H^+. \qquad (30)$$

Since the function

$$F(w) = \frac{w-1}{w+1}$$

maps conformally the right half-plane H^+ onto the unit disk **D**, we see that, by the Bieberbach formula (16), the function

$$T(w) = \log\frac{8|F'(w)|^2}{(1-|F(w)|^2)^2} = -2\log\operatorname{Re} w + \log 2$$

gives us a blow-up solution to Eq. (30) in H^+. Now, by formula (29), we have that the first required solution has the form

$$u(z) = T(\omega) = -2\log\operatorname{Re}\,\omega(z) + \log 2 = -2\log\operatorname{Re} x + \log 2.$$

The second solution can be obtained in the same way.

3. Free boundary. The effect of the "dead zone" very important for applications to solutions of some partial differential equations, see, e.g., [9], the Introduction and § 1, is that the solution of the corresponding differential equation vanishes on some nonempty open set of the domain of definition. For example, it is well known that the solution of the semilinear equation

$$\Delta u = u^q$$

may have the "dead zone" only when $0 < q < 1$, see, e.g., [9, p. 15].

We confine ourselves to only one result in this direction, which is again a simple consequence of Corollary 1.

Theorem 4. *Let* $\mathbf{C}$ *be the complex plane and let the matrix function*

$$A(z) = \begin{pmatrix} 1 & \mp\dfrac{2\nu(x)}{\sqrt{1-\nu^2(x)}} \\ \mp\dfrac{2\nu(x)}{\sqrt{1-\nu^2(x)}} & \dfrac{1+3\nu^2(x)}{1-\nu^2(x)} \end{pmatrix}, \tag{31}$$

where $\nu(x)$, $x \in \mathbf{R}$, *stands for an arbitrary measurable real-valued function such that* $|\nu(x)| \leqslant q < 1$. *Then the semilinear equation*

$$\operatorname{div}[A(z)\nabla u] = u^q, \quad 0 < q < 1, \quad z \in \mathbf{C}, \tag{32}$$

has the following solution with the "dead zone" in the complex plane:

$$u(x,y) = \begin{cases} \gamma\left(y \pm \displaystyle\int_0^x \frac{2\nu(t)}{\sqrt{1-\nu^2(t)}}\,dt\right)^{\frac{2}{1-q}}, & \text{if } y > \varphi(x), x \in \mathbf{R}, \\ 0 & \text{if } x \leqslant \varphi(x). \end{cases} \tag{33}$$

Here,

$$\gamma = \left(\frac{(1-q)^2}{2(1+q)} \right)^{\frac{1}{1-q}},$$

and

$$y = \varphi(x) = \pm \int_0^x \frac{2\nu(t)\,dt}{\sqrt{1-\nu^2(t)}}, \quad \infty < x < +\infty,$$

stands for the corresponding free boundary parametrization.

Indeed, the matrix function $A(z)$ generates a quasiconformal automorphism of the complex plane ω, which is expressed explicitly by formula (27). Since the mapping ω is volume-preserving, we can apply Corollary 1 in order to represent solutions to Eq. (32) in the form

$$u(z) = T(\omega(z)),$$

where $T(w)$ satisfies the equation

$$\Delta T(w) = T^q(w), \quad w = \xi + i\eta.$$

It remains to note that the function

$$T(w) = \gamma\eta^{\frac{2}{1-q}}, \quad \text{if} \quad \eta > 0$$

and $T(w) = 0$, if $\eta \leqslant 0$, satisfy the above equation.

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Received 07.12.2017

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ВИБУХОВІ РОЗВ'ЯЗКИ ТА МЕРТВІ ЗОНИ
ДЛЯ НАПІВЛІНІЙНИХ РІВНЯНЬ

Досліджено напівлінійне диференціальне рівняння виду $\mathrm{div}(A(z)\nabla u) = f(u)$ в $\Omega \subset \mathbf{C}$, де $A(z)$ — симетрична 2×2 матрична функція з вимірними коефіцієнтами, $\det A = 1$, і така, що $1/K|\xi|^2 \leqslant \langle A(z)\xi, \xi\rangle \leqslant K|\xi|^2, \xi \in \mathbf{R}^2$, $1 \leqslant K < \infty$. Із застосуванням теореми про факторизацію, доведену нами раніше, наведено явні розв'язки для зазначеного рівняння, якщо матриці $A(z)$ обрані належним чином і $f = e^u$ або $f = u^q$.

Ключові слова: *квазіконформні відображення, напівлінійні рівняння в частинних похідних, вибухові розв'язки.*

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О ВЗРЫВАЮЩИХСЯ РЕШЕНИЯХ И МЕРТВЫХ ЗОНАХ
ДЛЯ ПОЛУЛИНЕЙНЫХ УРАВНЕНИЙ

Исследовано полулинейное дифференциальное уравнение вида $\mathrm{div}(A(z)\nabla u) = f(u)$ в $\Omega \subset \mathbf{C}$, где $A(z)$ — симметричная 2×2 матричная функция с измеримыми коэффициентами, $\det A = 1$ и такая, что $1/K|\xi|^2 \leqslant \langle A(z)\xi, \xi\rangle \leqslant K|\xi|^2, \xi \in \mathbf{R}^2$, $1 \leqslant K < \infty$. С применением теоремы о факторизации, доказанной нами ранее, приведены явные решения для указанного уравнения, если матрицы $A(z)$ выбраны надлежащим образом и $f = e^u$ или $f = u^q$.

Ключевые слова: *квазиконформные отображения, полулинейные уравнения в частных производных, взрывающиеся решения.*