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On the Hilbert problem for analytic functions in quasihyperbolic domains

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We study the Hilbert boundary-value problem for analytic functions in the Jordan domains satisfying the quasihyperbolic boundary condition by Gehring—Martio. Assuming that the coefficients of the problem are functions of the countably bounded variation and the boundary data are measurable with respect to the logarithmic capacity, we prove the existence of solutions of the problem in terms of angular limits. As consequences, we derive the corresponding results concerning the Dirichlet, Neumann, and Poincaré boundary-value problems for harmonic functions.

Keywords: *Hilbert, Dirichlet, Neumann, and Poincaré boundary-value problems, analytic and harmonic functions, quasihyperbolic boundary condition, logarithmic capacity, angular limits.*

1. Introduction. D. Hilbert studied the boundary-value problem formulated as follows: To find an analytic function $f(z)$ in a domain D bounded by a rectifiable Jordan contour C that satisfies the boundary condition

$$\lim_{z\to\zeta} \operatorname{Re}\{\overline{\lambda(\zeta)} f(z)\} = \varphi(\zeta) \quad \forall \zeta \in C, \tag{1}$$

where both the *coefficient* λ and the *boundary data* φ of the problem are continuously differentiable with respect to the natural parameter s on C. Moreover, it was assumed by Hilbert that $\lambda \neq 0$ everywhere on C. The latter allows us, without loss of generality, to consider that $|\lambda| \equiv 1$ on C. In this case, the quantity $\operatorname{Re}\{\overline{\lambda} f\}$ from the left in (1) means a projection of f onto the direction λ interpreted as vectors in $\mathbb{R}^2$.

Historic surveys in the subject can be found in the recent papers [1-3]. Here, we substantially weaken the regularity conditions on the functions λ and φ in the boundary condition (1) and on the boundary C of the domain D. On the one hand, we will deal with the coefficients λ of *countably bounded variation* and the boundary data φ which are measurable with respect to *the loga-*

rithmic capacity. On the other hand, the fundamental Becker—Pommerenke result in [4] allows us to study the Hilbert boundary-value problem in domains D with the quasihyperbolic boundary condition introduced in [5].

Recall that the *quasihyperbolic distance* between points z and z_0 in a domain $D \subset \mathbb{C}$ is the quantity $k_D(z, z_0) := \inf\limits_{\gamma} \int\limits_{\gamma} ds/d(\zeta, \partial D)$, where $d(\zeta, \partial D)$ denotes the Euclidean distance from the point $\zeta \in D$ to ∂D, and the infimum is taken over all rectifiable curves γ joining the points z and z_0 in D, see [6]. Further, it is said that a domain D satisfies the *quasihyperbolic boundary condition* if, for constants a and b and a point $z_0 \in D$,

$$k_D(z, z_0) \leqslant a + b \ln \frac{d(z_0, \partial D)}{d(z, \partial D)} \quad \forall z \in D. \tag{2}$$

Let D be a Jordan domain in $\mathbb{C}$ such that it has a tangent at a point $\zeta \in \partial D$ A path in D terminating at ζ is called *nontangential* if its part in a neighborhood of ζ lies inside of an angle in D with the vertex at ζ. The limit along all nontangential paths at ζ is called *angular* at the point. The latter notion is a standard tool for the study of the boundary behavior of analytic and harmonic functions, see, e.g., [7-9]. Further, the Hilbert boundary condition (1) will be understood precisely in the sense of angular limit.

The notion of the logarithmic capacity is the important tool for our research. Dealing with measurable boundary data functions $\varphi(\zeta)$ with respect to the logarithmic capacity, see definitions in [3], we will use the *abbreviation q.e.* (*quasieverywhere*) on a set $E \subset \mathbb{C}$, if a property holds for all $\zeta \in E$ except its subset of zero logarithmic capacity [10].

2. Some definitions and preliminary remarks. Later on, $\mathbb{D}$ denotes the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Given a Jordan domain D in $\mathbb{C}$, we call $\lambda : \partial D \to \mathbb{C}$ a *function of bounded variation*, write $\lambda \in \mathcal{BV}(\partial D)$, if

$$V_{\lambda}(\partial D) := \sup \sum_{j=1}^{j=k} |\lambda(\zeta_{j+1}) - \lambda(\zeta_j)| < \infty \tag{3}$$

where the supremum is taken over all finite collections of points $\zeta_j \in \partial D$, $j = 1, \ldots, k$, with the cyclic order meaning that ζ_j lies between ζ_{j+1} and ζ_{j-1} for every $j = 1, \ldots, k$. Here, we assume that $\zeta_{k+1} = \zeta_1 = \zeta_0$. The quantity $V_{\lambda}(\partial D)$ is called the *variation of the function* λ.

Now, we call $\lambda : \partial D \to \mathbb{C}$ a function of the *countably bounded variation*, write $\lambda \in \mathcal{CBV}(\partial D)$, if there is a countable collection of mutually disjoint arcs γ_n of ∂D, $n = 1, 2, \ldots$ on each of which the restriction of λ is of bounded variation V_n, $\sup\limits_n V_n < \infty$, and the set $\partial D \setminus \cup \gamma_n$ has logarithmic capacity zero. In particular, the latter holds true if $\partial D \setminus \cup \gamma_n$ is countable. It is clear that such functions can be singular enough.

The following statement was proved as Proposition 5.1 in paper [3], where the function α_{λ} has been called by a *function of the argument* of λ.

Proposition 1. *For every function* $\lambda : \partial \mathbb{D} \to \partial \mathbb{D}$ *of the class* $\mathcal{BV}(\partial \mathbb{D})$, *there is a function* $\alpha_{\lambda} : \partial \mathbb{D} \to \mathbb{R}$ *of the class* $\mathcal{BV}(\partial \mathbb{D})$ *with* $V_{\alpha_{\lambda}} \leqslant V_{\lambda} 3\pi/2$ *such that* $\lambda(\zeta) = \exp\{i\alpha_{\lambda}(\zeta)\}$ *for all* $\zeta \in \partial \mathbb{D}$.

Given a Jordan curve $\Gamma \subset \mathbb{C}$, $L^{\infty}_c(\Gamma)$ denotes the class of all functions $\alpha : \Gamma \to \mathbb{R}$ which are measurable with respect to the logarithmic capacity and q.e. bounded.

Proposition 2. *For every function* $\lambda : \partial\mathbb{D} \to \partial\mathbb{D}$ *in the class* $\mathcal{CBV}(\partial\mathbb{D})$, *there is a function* $\alpha_\lambda : \partial\mathbb{D} \to \mathbb{R}$ *in the class* $L^\infty_c(\partial\mathbb{D}) \cap \mathcal{CBV}(\partial\mathbb{D})$ *such that*

$$\lambda(\zeta) = \exp\{i\alpha_\lambda(\zeta)\} \quad \textit{q.e. on } \partial\mathbb{D}. \tag{4}$$

Proof. Denote, by λ_n, the function on $\partial\mathbb{D}$ that is equal to λ on γ_n and to 1 outside of γ_n. Let α_n correspond to λ_n by Proposition 1. Then its variation $V^*_n \leqslant V_n 3\pi/2$. With no loss of generality, we may assume that $\alpha_n \equiv 0$ outside of γ_n. Set $\alpha = \sum\limits_{n=1}^{\infty} \alpha_n$. Then $\alpha \in \mathcal{CBV}(\partial\mathbb{D})$ and $\lambda(\zeta) = \exp\{i\alpha(\zeta)\}$ q.e. on $\partial\mathbb{D}$ Applying the corresponding shifts (divisible 2π), we may change α_n on γ_n through α^*_n with $|\alpha^*_n| \leqslant \pi$ at the middle point of γ_n. Then it is clear that the new function $\alpha^* \in \mathcal{CBV}(\partial\mathbb{D})$ and $\lambda(\zeta) = \exp\{i\alpha^*(\zeta)\}$ q.e. on $\partial\mathbb{D}$ and, moreover, $|\alpha^*| \leqslant \pi + V_n 3\pi/2$ on every γ_n, i.e $|\alpha^*|$ is bounded outside of the set $\partial\mathbb{D} \setminus \cup\gamma_n$. In addition, by construction, the function α^* is continuous q.e. on $\partial\mathbb{D}$ Hence, $\alpha^* \in L^\infty_c(\partial\mathbb{D})$.

We say that a Jordan curve Γ in $\mathbb{C}$ is *almost smooth* if Γ has a tangent quasieverywhere. Here, we say that a straight line L in $\mathbb{C}$ is *tangent* to Γ at a point $z_0 \in \Gamma$ if

$$\limsup_{z \to z_0,\, z \in \Gamma} \frac{dist(z, L)}{|z - z_0|} = 0. \tag{5}$$

In particular, Γ is almost smooth if Γ has a tangent at all its points except a countable set. The nature of such Jordan curves Γ is complicated enough because the countable set can be everywhere dense in Γ.

Remark 1. By Corollary of Theorem 1 in [4], a conformal mapping of a Jordan domain D in $\mathbb{C}$ with the quasihyperbolic boundary condition onto the unit disk $\mathbb{D}$ as well as its inverse are Hölder continuous in the closure of D and $\mathbb{D}$, respectively. Since the logarithmic capacity of a set coincides with its transfinite diameter, these mappings keep the sets of the logarithmic capacity zero on the boundaries of D and $\mathbb{D}$. Consequently, by Remark 2.1 in [3], such mappings also keep boundary functions which are measurable with respect to the logarithmic capacity. These facts are key for the research of the boundary-value problems in the given domains.

3. Correlation of conjugate harmonic functions. The following statement was first proven for the case of bounded variation in [3] as Theorem 5.1. Here, we give an alternative proof of this significant fact and extend it to the case of countably bounded variation.

Lemma 1. *Let* $\alpha : \partial\mathbb{D} \to \mathbb{R}$ *be in the class* $L^\infty_c(\partial\mathbb{D}) \cap \mathcal{CBV}(\partial\mathbb{D})$, *let* $u : \mathbb{D} \to \mathbb{R}$ *be a bounded harmonic function such that*

$$\lim_{z \to \varsigma} u(z) = \alpha(\zeta) \tag{6}$$

at every point of continuity of α, *and let* ν *be its conjugate harmonic function. Then* ν *has the angular limit*

$$\lim_{z \to \varsigma} \nu(z) = \beta(\zeta) \quad \textit{q.e. on } \partial\mathbb{D}, \tag{7}$$

where $\beta : \partial\mathbb{D} \to \mathbb{R}$ *is measurable with respect to the logarithmic capacity.*

Proof. Let us start from the case $\alpha\in \mathcal{BV}(\partial\mathbb{D})$. In this case, α has at most a countable set S of points of discontinuity and, consequently, S is of zero logarithmic capacity. Hence, by the generalized maximum principle, see the point 115 in [11], such a function u is unique and, thus, u can be represented as the Poisson integral of the function α, see Theorem I.D.2.2 in [8],

$$u(re^{i\vartheta})=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(\vartheta-t)+r^2}\alpha(e^{it})\,dt. \tag{8}$$

Here, the Poisson kernel is a real part of the analytic function $(\zeta+z)/(\zeta-z)$, $\zeta=e^{it}$, $z=re^{i\vartheta}$, and, by the Weierstrass theorem, see Theorem 1.1.1 in [12], the Schwartz integral

$$f(z):=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\alpha(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} \tag{9}$$

gives the analytic function $f=u+iv$ in $\mathbb{D}$ with $u=\operatorname{Re} f$, $v=\operatorname{Im} f$, and

$$f(z)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\alpha(e^{it})\frac{e^{it}+z}{e^{it}-z}dt=C+\frac{z}{\pi}\int\limits_{-\pi}^{\pi}\frac{F(t)}{1-e^{-it}z}dt, \tag{10}$$

where $F(t)=e^{-it}\alpha(e^{it})$ and $C=\dfrac{1}{2\pi}\displaystyle\int\limits_{-\pi}^{\pi}\alpha(e^{it})\,dt$. By Theorem 2(c) in [13], the function $f(z)$ has angular limits $f(\zeta)$ as $z\to\zeta$ q.e. on $\partial\mathbb{D}$, because the function F is of bounded variation. It remains to note that $f(\zeta)=\lim\limits_{n\to\infty}f_n(\zeta)$, where $f_n(\zeta)=f(r_n\zeta)$, for an arbitrary sequence $r_n\to 1-0$ as $n\to\infty$ q.e. on $\partial\mathbb{D}$. Thus, $f(\zeta)$ is measurable with respect to the logarithmic capacity, because the functions $f_n(\zeta)$ are so as continuous functions on $\partial\mathbb{D}$, see 2.3.10 in [14].

Now, let $\alpha\in \mathcal{CBV}(\partial\mathbb{D})$. Then its set of points of discontinuity is at most of zero logarithmic capacity. Hence, again by the generalized maximum principle, the bounded function u satisfying (6) is unique. Moreover, $\alpha\in L^{\infty}_{c}(\partial\mathbb{D})$ and, consequently, u can be represented by the Poisson integral (8), and the Schwartz integral (9) gives the analytic function $f=u+iv$ in $\mathbb{D}$, where

$$v(re^{i\vartheta})=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{2r\sin(\vartheta-t)}{1-2r\cos(\vartheta-t)+r^2}\alpha(e^{it})\,dt\,. \tag{11}$$

Let us apply the linearity of the integral operator (11). Namely, denote, by χ, the characteristic function of an arc γ_* of $\partial\mathbb{D}$, where α is of bounded variation from the definition of $\mathcal{CBV}$. Setting $\alpha_*=\alpha\chi$ and $\alpha_0=\alpha-\alpha_*$, we have $\alpha=\alpha_*+\alpha_0$. Then $\nu=\nu_*+\nu_0$ where ν_* and ν_0 correspond to α_* and α_0 by formula (11). By the first item of the proof, there exists the angular limit $\lim\limits_{z\to\zeta}v_*(z)=\beta_*(\zeta)$ q.e. on $\partial\mathbb{D}$, where $\beta_*:\partial\mathbb{D}\to\mathbb{R}$ is a measurable function with respect to the logarithmic capacity. Moreover, it is evident from formula (11) that $v_0(z)\to\beta_0(\zeta)$ as $z\to\zeta$ for all $\zeta\in\gamma_*$, where $\beta_0:\gamma_*\to\mathbb{R}$ is continuous on γ_*. Thus, setting $\beta=\beta_*+\beta_0$ on γ_*, we obtain the conclusion of Lemma 1 because the collection of such arcs γ_* is countable, and the completion of this collection on $\partial\mathbb{D}$ has zero logarithmic capacity.

4. The Hilbert problem for analytic functions.

Theorem 1. *Let* $\lambda:\partial\mathbb{D}\to\partial\mathbb{D}$ *be in the class* $\mathcal{CBV}(\partial\mathbb{D})$ *and* $\varphi:\partial\mathbb{D}\to\mathbb{R}$ *be measurable with respect to the logarithmic capacity. Then there is an analytic function* $f:\mathbb{D}\to\mathbb{C}$ *with the angular limit*

$$\lim_{z\to\zeta}\operatorname{Re}[\overline{\lambda(\zeta)}f(z)]=\varphi(\zeta)\quad q.e.\ on\ \partial\mathbb{D}. \tag{12}$$

Proof. By Proposition 2, the function $\alpha_\lambda\in L^\infty_c(\partial\mathbb{D})\cap\mathcal{CBV}(\partial\mathbb{D})$. Therefore,

$$g(z):=\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}}\alpha_\lambda(\zeta)\frac{z+\zeta}{z-\zeta}\frac{d\zeta}{\zeta},\quad z\in\mathbb{D},$$

is an analytic function with $u(z)=\operatorname{Re}g(z)\to\alpha_\lambda(\zeta)$ as $z\to\zeta$ for every $\zeta\in\partial\mathbb{D}$ except a set of the discontinuity points for the function α_λ, which has zero logarithmic capacity, see Corollary IX.1.1 in [12] and Theorem I.D.2.2 in [8]. Note that the function $A(z)=\exp\{ig(z)\}$ is also analytic. By Lemma 1, there is a function $\beta:\partial\mathbb{D}\to\mathbb{R}$ that has the angular limit $\nu(z)=\operatorname{Im}g(z)\to$ $\to\beta(\zeta)$ as $z\to\zeta$ q.e. on $\partial\mathbb{D}$ and β is measurable with respect to the logarithmic capacity. Thus, by Corollary 4.1 in [3], there exists an analytic function $B:\mathbb{D}\to\mathbb{C}$ that has the angular limit $U(z)=\operatorname{Re}B(z)\to\varphi(\zeta)\exp\{\beta(\zeta)\}$ as $z\to\zeta$ q.e. on $\partial\mathbb{D}$ Finally, an elementary computation shows that the desired function has the form $f=AB$.

Theorem 2. *Let* D *be a Jordan domain with the quasihyperbolic boundary condition, let* ∂D *have a tangent q.e., let* $\lambda:\partial D\to\mathbb{C},|\lambda(\zeta)|\equiv 1$, *be in* $\mathcal{CBV}(\partial D)$, *and let* $\varphi:\partial D\to\mathbb{R}$ *be measurable with respect to the logarithmic capacity. Then there is an analytic function* $f:D\to\mathbb{C}$ *with the angular limit*

$$\lim_{z\to\zeta}\operatorname{Re}[\overline{\lambda(\zeta)}f(z)]=\varphi(\zeta)\quad q.e.\ on\ \partial D \tag{13}$$

Proof. Let g be a conformal mapping of D onto $\mathbb{D}$ that exists by the Riemann mapping theorem, see Theorem II.2.1 in [12], and by the Caratheodory theorem, see Theorem II.3.4 in [12], g be extended to a homeomorphism $\tilde g$ of $\overline{D}$ onto $\overline{\mathbb{D}}$. By Corollary of Theorem 1 in [4], $g_*:=\tilde g|_{\partial D}$ and its inverse function are Hölder continuous. Then $\Lambda:=\lambda\circ g_*^{-1}\in\mathcal{CBV}(\partial\mathbb{D})$ and $\Phi:=\varphi\circ g_*^{-1}$ is measurable with respect to the logarithmic capacity by Remark 1. Thus, by Theorem 1, there is an analytic function $A:\mathbb{D}\to\mathbb{C}$ that has the angular limit

$$\lim_{\omega\to\eta}\operatorname{Re}\{\overline{\Lambda(\eta)}A(\omega)\}=\Phi(\eta)\quad q.e.\ on\ \partial\mathbb{D}. \tag{14}$$

Let us consider the analytic function $f:=A\circ g$ and show that f is desired. Indeed, by the Lindelöf theorem, see Theorem II.C.2 in [8], if ∂D has a tangent at a point ζ, then $\arg[g(\zeta)-g(z)]-\arg[\zeta-z]\to\text{const}$ as $z\to\zeta$. In other words, the images under the conformal mapping g of sectors in D with a vertex at ζ is asymptotically the same as sectors in $\mathbb{D}$ with a vertex at $w=g(\zeta)$. Consequently, nontangential paths in D are transformed under g into nontangential paths in $\mathbb{D}$ and inversely q.e. on ∂D and $\partial\mathbb{D}$ respectively, because D is almost smooth and g_* and g_*^{-1} keep sets of logarithmic capacity zero. Thus, (14) implies the existence of the angular limit (13) q.e. on ∂D.

5. On Dirichlet, Neumann, and Poincaré problems. We reduce these boundary-value problems for harmonic functions to suitable Hilbert problems for analytic functions studied above.

Corollary 1. *Let D be a Jordan domain with the quasihyperbolic boundary condition and let ∂D have a tangent q.e. Suppose $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the logarithmic capacity. Then there exists a harmonic function $u : D \to \mathbb{C}$ that has the angular limit*

$$\lim_{z \to \zeta} u(z) = \varphi(\zeta) \quad \text{q.e. on } \partial D\,. \tag{15}$$

It is well known that the Neumann problem, in general, has no classical solution. The necessary condition of solvability is that the integral of the function φ over $\partial\mathbb{D}$ is equal to zero [15].

Theorem 3. *Let D be a Jordan domain with the quasihyperbolic boundary condition and let ∂D have a tangent q.e. Suppose that $\nu : \partial D \to \mathbb{C}, |\nu(\zeta)| \equiv 1$, is in the class $\mathcal{CBV}$ and $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the logarithmic capacity. Then there exists a harmonic function $u : D \to \mathbb{R}$ with the angular limit*

$$\lim_{z \to \zeta} \frac{\partial u}{\partial \nu} = \varphi(\zeta) \quad \text{q.e. on } \partial D\,. \tag{16}$$

Proof. Indeed, by Theorem 2, there exists an analytic function $f : D \to \mathbb{C}$ that has the angular limit

$$\lim_{z \to \zeta} \operatorname{Re}[\nu(\zeta) f(z)] = \varphi(\zeta) \tag{17}$$

q.e. on ∂D. Note that an indefinite integral F of f in D is also an analytic function and, correspondingly, the harmonic functions $u = \operatorname{Re} F$ and $v = \operatorname{Im} F$ satisfy the Cauchy—Riemann system $v_x = -u_y$ and $v_y = u_x$. Hence $f = F' = F_x = u_x + iv_x = u_x - iu_y = \overline{\nabla u}$ where $\nabla u = u_x + iu_y$ is the gradient of the function u in the complex form. Thus, (16) follows from (17), i.e. u is the desired harmonic function, because its directional derivative $\dfrac{\partial u}{\partial \nu} = \operatorname{Re} \overline{\nu} \nabla u = \operatorname{Re} \nu \overline{\nabla u} = \langle \nu, \nabla u \rangle$ is the scalar product of ν and the gradient ∇u.

Corollary 2. *Let D be a Jordan domain in C with the quasihyperbolic boundary condition and let the unit interior normal $n(\zeta)$ to the boundary ∂D be in the class $\mathcal{CBV}$. Suppose that $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the logarithmic capacity. Then one can find a harmonic function $u : D \to \mathbb{C}$ such that q.e. on ∂D there exist:*

1) *the finite limit along the normal $n(\zeta)$*

$$u(\zeta) := \lim_{z \to \zeta} u(z)\,,$$

2) *the normal derivative*

$$\frac{\partial u}{\partial n}(\zeta) := \lim_{t \to 0} \frac{u(\zeta + tn) - u(\zeta)}{t} = \varphi(\zeta)\,,$$

3) *the angular limit*

$$\lim_{z \to \zeta} \frac{\partial u}{\partial n}(z) = \frac{\partial u}{\partial n} = (\zeta)$$

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ПРО ЗАДАЧУ ГІЛЬБЕРТА ДЛЯ АНАЛІТИЧНИХ ФУНКЦІЙ
У КВАЗІГІПЕРБОЛІЧНИХ ОБЛАСТЯХ

Досліджено граничну задачу Гільберта для аналітичних функцій в жорданових областях, які задовольняють квазігіперболічну умову Герінга—Мартіо. З припущенням, що коефіцієнти задачі є функціями зліченно-обмеженої варіації і граничні дані є вимірними відносно логарифмічної ємності, доведено існування розв'язків задачі в термінах кутових границь. Як наслідки отримано відповідні результати для крайових задач Діріхле, Неймана і Пуанкаре для гармонічних функцій.

Ключові слова: *крайові задачі Гільберта, Діріхле, Неймана і Пуанкаре, аналітичні і гармонічні функції, квазігіперболічна гранична умова, логарифмічна ємність, кутова границя.*

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О ЗАДАЧЕ ГИЛЬБЕРТА ДЛЯ АНАЛИТИЧЕСКИХ ФУНКЦИЙ В КВАЗИГИПЕРБОЛИЧЕСКИХ ОБЛАСТЯХ

Исследована краевая задача Гильберта для аналитических функций в жордановых областях, удовлетворяющих квазигиперболическому условию Геринга—Мартио. С предположением, что коэффициенты задачи являются функциями счетно-ограниченной вариации, а граничные данные измеримы относительно логарифмической емкости, доказано существование решений задачи в терминах угловых пределов. В качестве следствий получены соответствующие результаты для краевых задач Дирихле, Неймана и Пуанкаре для гармонических функций.

Ключевые слова: *краевые задачи Гильберта, Дирихле, Неймана и Пуанкаре, аналитические и гармонические функции, квазигиперболическое краевое условие, логарифмическая емкость, угловой предел.*