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A.O. Borysyuk, Corresponding Member of the NAS of Ukraine, https://orcid.org/0000-0002-3878-3915
Institute of Hydromechanics of the NAS of Ukraine, Kyiv
E-mail: aobor@ukr.net

A numerical technique to solve a problem of the fluid motion in a straight plane rigid duct with two axisymmetric rectangular constrictions. An alternative approach

A numerical technique is devised to solve a problem of the steady laminar fluid motion in a straight plane hard-walled duct with two local axisymmetric rectangular constrictions. It uses the stream function, the vorticity and the pressure as the basic variables, has the second order of accuracy in the spatial and the first order of accuracy in the temporal coordinates, provides high stability of a solution, and needs significantly less computational time to obtain a result compared to appropriate techniques available in a scientific literature. The technique consists in: a) introducing the stream function and the vorticity, and subsequent transiting from the non-dimensional governing relations for the fluid velocity and the pressure to the corresponding non-dimensional relations for the stream function, the vorticity and the pressure; b) deriving their discrete counterparts in the nodes of the chosen space-time computational grid; c) integrating the systems of linear algebraic equations obtained after making the discretization. The discretization is based on applying appropriate differencing schemes to the terms of the equations for the basic variables. These are the two-point temporal onward difference for the unsteady term of the vorticity equation, as well as the two-point backward differences (for its convective term) and the five-point approximations (for its diffusive term and for the Poisson's equations for the stream function and the pressure) in the axial and cross-flow coordinates. As for the velocity components, the appropriate central differences are applied to discretize their expressions. The above-mentioned systems of linear algebraic equations for the stream function and the pressure are integrated by the iterative successive over-relaxation method. The algebraic relation for the vorticity does not need application of any method to be solved, because it is a computational scheme to find this quantity based on the known magnitudes computed at the previous instant of time.

Keywords: *fluid motion, plane duct, rectangular constriction, technique.*

Study of fluid motions in ducts is an actual problem in medicine, architecture, municipal economy, etc. Among others, here a significant interest is shown for investigating flows in ducts with local constrictions, such as wall deposits, welding joints, stenoses, etc. That is due to the fact that such inhomogeneities in the duct geometry cause local changes in the flow structure and/or character, etc., and those changes can result in the corresponding consequences not only in the vicinity of, but also far from the inhomogeneities [1].

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As analysis of the scientific literature shows, study of flows in ducts with local constrictions has been paid much attention to. In those studies, straight rigid ducts and their constrictions of the simplest geometries were of interest. The basic flow (i. e., the flow upstream of a (first) constriction) was laminar, axisymmetric and steady. As for fluids, they were homogeneous, incompressible and Newtonian (the other types of ducts, their constrictions, fluids and the basic flow are not considered in this paper, because they were studied much less intensively compared with the noted ones). These allowed one, on the one hand, to investigate (within the framework of appropriate models chosen and with acceptable accuracy) the influence of the basic parameters of a duct, its constriction(s) and the basic flow on the flow not only near but also far downstream of the constriction(s), and, on the other hand, to simplify significantly solutions to the corresponding problems of interest [1-4].

Among the results obtained in those investigations, numerical techniques, which allow studying the fluid motion near duct constrictions, are of a great importance. Recently, a technique of such kind was described in [1]. It has been developed to investigate the flow in a straight flat rigid duct having two rectangular axisymmetric constrictions. That technique has a second order of accuracy both in the temporal and spatial co-ordinates, and allows making appropriate investigation in the variables velocity-pressure. However, due to the huge amount of mathematical operations, it needs a lot of computational time to obtain a result.

In this work, an alternative technique is described to solve the same problem. It uses the stream function, the vorticity and the pressure as the basic variables, and has nearly the same order of accuracy as that noted above. However, this technique has higher stability of a solution and, due to the use of less powerful mathematical apparatus, needs much less computational time to obtain a result.

Statement of the problem. A detailed statement of the initial and boundary problem to be solved, is available in [1]. Proceed from this, now we shall only briefly remind it.

A straight flat rigid duct of dimensionless width 1 is considered (Fig.1) in which fluid moves. In this duct, two narrowed axisymmetric segments of a rectangular form are available. They have the cross-flow, D_i, and axial, L_i, dimensions ($i=1,2$), respectively, and are separated by the distance L_{12} from one another. The fluid is viscous Newtonian and homogeneous, and has mass density ρ and kinematic viscosity ν. Its flow is characterised by a small Mach number and the rate Q per unit depth of the duct. Also, the flow upstream of the first narrowed segment (i. e., the basic flow) is steady and laminar. It is necessary to investigate the fluid motion around the segments.

The fluid motion is governed by the dimensionless momentum and continuity equations, viz.

$$\frac{\partial U_i}{\partial T}+U_j\frac{\partial U_i}{\partial X_j}=-\frac{\partial P}{\partial X_i}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial X_j}\left(\frac{\partial U_i}{\partial X_j}\right),\quad i=1,2, \tag{1}$$

$$\partial U_i/\partial X_i=0. \tag{2}$$

The boundary conditions are in the absence of a fluid motion at the duct wall, S_{ch}, and on both its narrowed segments, S_j ($j=1,2$). Also, since the mass conservation in the duct is assumed, the flow rate Q must not vary along the duct, viz.

$$U_i\big|_{S_{ch},S_j}=0,\quad \partial Q/\partial X_1=0,\quad Q=1,\quad i,j=1,2. \tag{3}$$

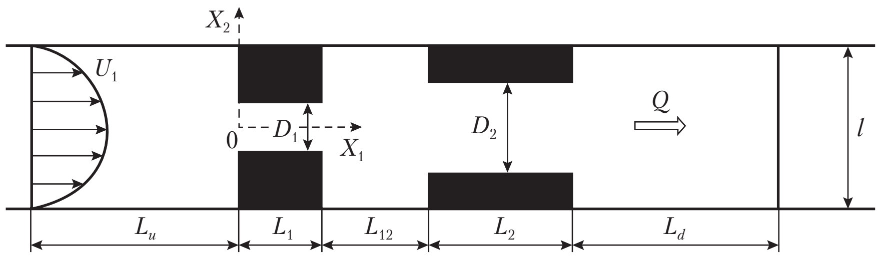

Fig. 1. Geometry of the problem and the computational domain

Apart from these, due to the laminar basic flow regime, the parabolic velocity profile is specified outside the disturbed flow region due to the narrowed segments[1], viz.

$$U_1\big|_{X_1=-L_u,\,L_1+L_{12}+L_2+L_d} = 1.5(1-4X_2^2)\,, \quad U_2\big|_{X_1=-L_u,\,L_1+L_{12}+L_2+L_d} = 0\,. \tag{4}$$

As for the pressure P, it is assumed to be constant both sufficiently far upstream of the first narrowing ($P\big|_{X_1=-L_u} = P_u$), and far downstream of the second one ($P\big|_{X_1=L_1+L_{12}+L_2+L_d} = P_d$). In addition, the difference $\Delta P = P_u - P_d > 0$ should ensure the existence of the given laminar regime of the basic flow. Also, without loss of generality, the pressure P_d can be taken to be zero[2], and the magnitude P_u, like the pressure in the whole duct, should be found. Apart from these, the normal pressure derivative is zero on the rigid walls of the duct and both its narrowed segments, viz.

$$(\partial P / \partial n)_{S_{ch},S_j} = 0\,, \quad j = 1, 2\,. \tag{5}$$

Regarding the initial conditions, they are in absence of the fluid motion in the duct at the time instant $T = 0$ [1], viz.

$$P\big|_{T=0} = 0\,, \quad U_i\big|_{T=0} = 0\,. \tag{6}$$

In (1)-(6), X_i ($i = 1, 2, 3$) are the rectangular Cartesian coordinates depicted in Fig. 1 (here the axis is normal to the plane X_1X_2 and directed to us); T the time; U_i the local fluid velocities in the directions X_i; $\mathrm{Re} = U_a D / \nu$ the Reynolds number of the cross-sectionally averaged basic flow; U_a its velocity; and the values of the distances L_u and L_d are given in section "Computational domain and computational mesh". In addition, hereinafter the vector $\vec{n}$ denotes the outward unit normal to appropriate surface, and the summation on repeated indices is assumed throughout the paper. As for the scaling factors used in this paper, these are the duct width D as the length scale, the velocity U_a as the velocity scale, the product $U_a D$ as the scale for both the flow rate and the stream function, the ratios D / U_a and U_a / D as the time scale and the vorticity scale, respectively, and the product ρU_a^2 as the pressure scale (the stream function and the vorticity are introduced in the next section).

[1] This is the region before the segments, where the flow is still undisturbed by them, and far behind them, where the flow is already undisturbed (i.e., where the flow disturbances disappear, and it becomes like the basic one).

[2] A choice of the value of P_d always can be compensated for by the choice of the corresponding value of P_u in such a way that the corresponding pressure drop ΔP (which governs fluid motion in the duct) remains unchangeable.

Transition to the variables stream function-vorticity-pressure. We begin to solve the problem formulated in the previous section with introducing the stream function Ψ (which satisfies (2)) and the vorticity Ω, viz.

$$U_1 = \frac{\partial \Psi}{\partial X_2}, \quad U_2 = -\frac{\partial \Psi}{\partial X_1}, \quad \Omega = \frac{\partial U_2}{\partial X_1} - \frac{\partial U_1}{\partial X_2}. \tag{7}$$

This allows one to rewrite the initial and boundary problem (1)-(6) in the variables Ψ, Ω and P instead of U_i, P. More specifically, making the differentiation of the equations for U_1 and U_2 in (1) with respect to X_2 and X_1, respectively, subsequent taking the difference of the second and the first of the obtained relationships and using the third relation in (7) yields the vorticity transfer equation, viz.

$$\frac{\partial \Omega}{\partial T} + U_1 \frac{\partial \Omega}{\partial X_1} + U_2 \frac{\partial \Omega}{\partial X_2} = \frac{1}{\mathrm{Re}} \nabla^2_{(X_1, X_2)} \Omega, \tag{8}$$

in which

$$\nabla^2_{(X_1, X_2)} = \frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2}$$

is the Laplace operator in the plane X_1, X_2.

Further use of the first two representations in (7) in the third one gives one the Poisson's equation for the stream function, viz.

$$\nabla^2_{(X_1, X_2)} \Psi = -\Omega. \tag{9}$$

Taking the derivatives of the equations for U_1 and U_2 in (1) with respect to X_1 and X_2, respectively, making the summation of the obtained relations and the use of the continuity equation (2) results in the Poisson's equation for the pressure, viz.

$$\nabla^2_{(X_1, X_2)} P = -\left(\frac{\partial U_1}{\partial X_1}\right)^2 - 2\frac{\partial U_1}{\partial X_2}\frac{\partial U_2}{\partial X_1} - \left(\frac{\partial U_2}{\partial X_2}\right)^2. \tag{10}$$

Regarding the boundary and initial conditions for the quantities Ψ, Ω and P, they are easily derived from (3)-(6). Indeed, based on (4) and (7) one can obtain the following conditions for Ψ and Ω on the upper, $X_1 = -L_u$, and lower, $X_1 = L_1 + L_{12} + L_2 + L_d$ (in the flow direction) boundaries of the disturbed flow region due to the narrowed segments, viz.

$$\Psi\big|_{X_1 = -L_u,\, L_1 + L_{12} + L_2 + L_d} = \frac{3}{2} X_2 \left(1 - \frac{4}{3} X_2^2\right), \quad \Omega\big|_{X_1 = -L_u,\, L_1 + L_{12} + L_2 + L_d} = 12 X_2. \tag{11}$$

A zero normal component of the fluid velocity on the walls of the duct and both its narrowed segments (see (3)) together with the first two representations in (7) give one the corresponding constancy of the stream function, viz.

$$\Psi\big|_{S_{ch}^{+},\,S_j^{+}} = \mathrm{const}_{+}\,,\quad \Psi\big|_{S_{ch}^{-},\,S_j^{-}} = \mathrm{const}_{-}\,,\quad j=1,2$$

(here the index "+" indicates the upper walls of the duct and the segments, whereas "–" their lower walls). This constancy together with the first relation in (11) yields

$$\Psi\big|_{S_{ch}^{+},\,S_j^{+}} = \frac{1}{2}\,,\quad \Psi\big|_{S_{ch}^{-},\,S_j^{-}} = -\frac{1}{2}\,,\quad j=1,2\,. \tag{12}$$

The absence of the tangential fluid motion on S_{ch} and S_j (see (3)) results in absence of the first-order normal derivatives and the second-order mixed derivative of the function Ψ there, viz.

$$\frac{\partial\Psi}{\partial X_2}\bigg|_{S_{ch},\,S_j^{\mathrm{h}}} = 0\,,\quad \frac{\partial\Psi}{\partial X_1}\bigg|_{S_j^{\mathrm{v}}} = 0\,,\quad \frac{\partial^2\Psi}{\partial X_1\partial X_2}\bigg|_{S_{ch},\,S_j} = 0\,,\quad j=1,2 \tag{13}$$

(in (13), S_j^{h} and S_j^{v} are the horizontal and vertical parts of S_j, respectively).

Relations (13) together with (9) provide the following boundary conditions for the vorticity

$$\Omega\big|_{S_{ch},\,S_j^{\mathrm{h}}} = -\frac{\partial^2\Psi}{\partial X_2^2}\bigg|_{S_{ch},\,S_j^{\mathrm{h}}}\,,\quad \Omega\big|_{S_j^{\mathrm{v}}} = -\frac{\partial^2\Psi}{\partial X_1^2}\bigg|_{S_j^{\mathrm{v}}}\,,\quad j=1,2\,. \tag{14}$$

As for boundary conditions for the pressure, their set (5) is completed with the following two more ones

$$\frac{\partial P}{\partial X_1}\bigg|_{S_j^{\mathrm{v}}} = \left[\frac{1}{\mathrm{Re}}\nabla^2_{(X_1,X_2)}U_1 - \frac{\partial U_1}{\partial T}\right]_{S_j^{\mathrm{v}}}\,,\quad \frac{\partial P}{\partial X_2}\bigg|_{S_{ch},\,S_j^{\mathrm{h}}} = \left[\frac{1}{\mathrm{Re}}\nabla^2_{(X_1,X_2)}U_2 - \frac{\partial U_2}{\partial T}\right]_{S_{ch},\,S_j^{\mathrm{h}}}\,, \tag{15}$$

which are obtained from (1) and the first condition in (3).

Finally, the initial conditions for Ψ, Ω and P are derived from (6) after the use of (7) there, viz.

$$\Psi\big|_{T=0} = 0\,,\quad \Omega\big|_{T=0} = 0\,,\quad P\big|_{T=0} = 0\,. \tag{16}$$

Computational domain and computational mesh. The domain, in which a solution to the problem should be found, is bounded by the duct sections $X_1 = -L_u$ and $X_1 = L_1 + L_{12} + L_2 + L_d$ (see Fig. 1). The first of them is chosen before the first narrowed segment, where the flow is still undisturbed by it, whereas the second section downstream of the second segment, where the flow redevelops into the basic state at $X_1 = -L_u$. Herewith, for the basic flow velocities considered in this study, the values of L_u and L_d should not exceed 0.5 and 12, respectively, viz. $L_u \leqslant 0.5$ and $L_d \leqslant 12$ [1, 2].

In this domain, one introduces a uniform rectangular computational mesh with the small steps Δ_i in the directions X_i ($i = 1, 2$, Fig. 2), viz.

$$X_{1n} = X_{1(n-1)} + \Delta_1\,,\quad X_{2m} = X_{2(m-1)} + \Delta_2\,,\quad \Delta_i = \mathrm{const}_i << 1\,. \tag{17}$$

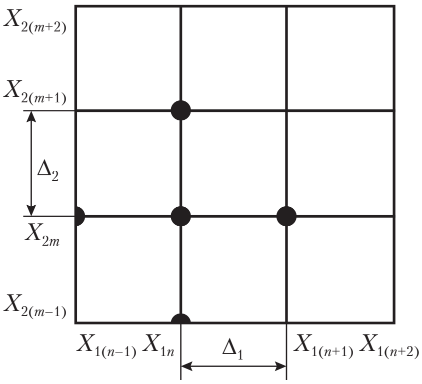

Fig. 2. Computational mesh

Regarding the integration time, it is divided into the small constant intervals Δ_T , viz.

$$T_k = T_{k-1} + \Delta_T = k\Delta_T \,, \quad \Delta_T = \mathrm{const}_T << 1 \,, \quad T_0 = 0 \,. \tag{18}$$

After that, one performs a discretization of the appropriate relationships of the previous subsection at the nodes X_{1n}, X_{2m}, T_k of the grid (17), (18). In making this, one denotes the values of an arbitrary quantity f at the point X_{1n}, X_{2m}, T_k by $f_{n,m}^k$, viz.

$$f_{n,m}^k = f(X_1, X_2, T)\big|_{X_1 = X_{1n},\, X_2 = X_{2m},\, T = T_k} \,.$$

Discrete forms of the equations for Ψ , Ω and P . Discrete forms of the first two relations in (7) are resulted from the use of the appropriate central differences [5], viz.

$$(U_1)_{n,m}^k = \frac{\Psi_{n,m+1}^k - \Psi_{n,m-1}^k}{2\Delta_2} \,, \quad (U_2)_{n,m}^k = -\frac{\Psi_{n+1,m}^k - \Psi_{n-1,m}^k}{2\Delta_1} \,. \tag{19}$$

Expressions (19) are of the second order of accuracy.

A discrete analog of the third relation in (7) is not given here, since it is not used in this article.

A discretization of the vorticity transfer equation (8) is performed on the basis of the appropriate differencing schemes. More specifically, the non-steady term in (8) is discretized on the basis of the two-point temporal onward difference, viz.

$$\left(\frac{\partial \Omega}{\partial T}\right)_{n,m}^k = \frac{\Omega_{n,m}^{k+1} - \Omega_{n,m}^k}{\Delta_T} \,. \tag{20}$$

Representation (20) has the first order of accuracy.

The application of the two-point backward difference schemes in the coordinates X_1 and X_2 to the convective term of Eq. (8) yields its discrete counterpart of the second order of accuracy, viz.

$$\begin{gathered}
\left(U_1 \frac{\partial \Omega}{\partial X_1}\right)_{n,m}^k = \begin{cases} (U_1)_{n,m}^k \dfrac{\Omega_{n,m}^k - \Omega_{n-1,m}^k}{\Delta_1}; \ (U_1)_{n,m}^k \geqslant 0, \\[2ex] (U_1)_{n,m}^k \dfrac{\Omega_{n+1,m}^k - \Omega_{n,m}^k}{\Delta_1}; \ (U_1)_{n,m}^k < 0, \end{cases} \\
\left(U_2 \frac{\partial \Omega}{\partial X_2}\right)_{n,m}^k = \begin{cases} (U_2)_{n,m}^k \dfrac{\Omega_{n,m}^k - \Omega_{n,m-1}^k}{\Delta_2}; \ (U_2)_{n,m}^k \geqslant 0, \\[2ex] (U_2)_{n,m}^k \dfrac{\Omega_{n,m+1}^k - \Omega_{n,m}^k}{\Delta_2}; \ (U_2)_{n,m}^k < 0. \end{cases}
\end{gathered} \tag{21}$$

And the last, the diffusive term of Eq. (8) has the following discrete form

$$\left(\frac{\partial^2\Omega}{\partial X_1^2}\right)_{n,m}^k = \frac{\Omega_{n+1,m}^k - 2\Omega_{n,m}^k + \Omega_{n-1,m}^k}{\Delta_1^2}, \quad \left(\frac{\partial^2\Omega}{\partial X_2^2}\right)_{n,m}^k = \frac{\Omega_{n,m+1}^k - 2\Omega_{n,m}^k + \Omega_{n,m-1}^k}{\Delta_2^2}. \tag{22}$$

Expressions (22) are of the second order of accuracy and derived on the basis of the five-point differencing scheme (see Fig. 2).

Relations (20)-(22) together with (19) give one a discrete analog of Eq. (8), viz.

$$\Omega_{n,m}^{k+1} = C_{n,m}^k\Omega_{n,m}^k + C_{n-1,m}^k\Omega_{n-1,m}^k + C_{n+1,m}^k\Omega_{n+1,m}^k + C_{n,m-1}^k\Omega_{n,m-1}^k + C_{n,m+1}^k\Omega_{n,m+1}^k. \tag{23}$$

Here the factors $C_{\dots}^{\dots}$ are written as

$$C_{n,m}^k = \begin{cases} 1-\alpha_{12}\left((\Delta\Psi)_{n,m,2}^k - (\Delta\Psi)_{n,m,1}^k\right) - 2\alpha_1 - 2\alpha_2; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k \geqslant 0; \\ 1-\alpha_{12}\left((\Delta\Psi)_{n,m,2}^k + (\Delta\Psi)_{n,m,1}^k\right) - 2\alpha_1 - 2\alpha_2; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k < 0; \\ 1+\alpha_{12}\left((\Delta\Psi)_{n,m,2}^k + (\Delta\Psi)_{n,m,1}^k\right) - 2\alpha_1 - 2\alpha_2; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k \geqslant 0; \\ 1+\alpha_{12}\left((\Delta\Psi)_{n,m,2}^k - (\Delta\Psi)_{n,m,1}^k\right) - 2\alpha_1 - 2\alpha_2; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k < 0; \end{cases}$$

$$C_{n-1,m}^k = \begin{cases} \alpha_{12}(\Delta\Psi)_{n,m,2}^k + \alpha_1; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k \geqslant 0; \\ \alpha_{12}(\Delta\Psi)_{n,m,2}^k + \alpha_1; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k < 0; \\ \alpha_1; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k \geqslant 0; \\ \alpha_1; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k < 0; \end{cases}$$

$$C_{n+1,m}^k = \begin{cases} \alpha_1; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k \geqslant 0; \\ \alpha_1; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k < 0; \\ -\alpha_{12}(\Delta\Psi)_{n,m,2}^k + \alpha_1; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k \geqslant 0; \\ -\alpha_{12}(\Delta\Psi)_{n,m,2}^k + \alpha_1; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k < 0; \end{cases}$$

$$C_{n,m-1}^k = \begin{cases} -\alpha_{12}(\Delta\Psi)_{n,m,1}^k + \alpha_2; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k \geqslant 0; \\ \alpha_2; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k < 0; \\ -\alpha_{12}(\Delta\Psi)_{n,m,1}^k + \alpha_2; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k \geqslant 0; \\ \alpha_2; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k < 0; \end{cases}$$

$$C_{n,m+1}^k = \begin{cases} \alpha_2; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k \geqslant 0; \\ \alpha_{12}(\Delta\Psi)_{n,m,1}^k + \alpha_2; \ (U_1)_{n,m}^k \geqslant 0, \ (U_2)_{n,m}^k < 0; \\ \alpha_2; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k \geqslant 0; \\ \alpha_{12}(\Delta\Psi)_{n,m,1}^k + \alpha_2; \ (U_1)_{n,m}^k < 0, \ (U_2)_{n,m}^k < 0; \end{cases}$$

in which the quantities α_1, α_2, α_{12} are the ratios of the steps of the computational grid (17), (18), viz.

$$\alpha_1 = \frac{1}{\mathrm{Re}}\frac{\Delta_T}{\Delta_1^2}, \quad \alpha_2 = \frac{1}{\mathrm{Re}}\frac{\Delta_T}{\Delta_2^2}, \quad \alpha_{12} = \frac{\Delta_T}{2\Delta_1\Delta_2},$$

and $(\Delta\Psi)^k_{n,m,i}$ the increase of the stream function in the direction X_i of the grid ($i = 1, 2$), viz.

$$(\Delta\Psi)^k_{n,m,1} = \Psi^k_{n+1,m} - \Psi^k_{n-1,m}, \quad (\Delta\Psi)^k_{n,m,2} = \Psi^k_{n,m+1} - \Psi^k_{n,m-1}.$$

Regarding the discrete forms of the Poisson's equations (9) and (10), they are similar, viz.

$$\frac{\Psi^k_{n+1,m} - 2\Psi^k_{n,m} + \Psi^k_{n-1,m}}{\Delta_1^2} + \frac{\Psi^k_{n,m+1} - 2\Psi^k_{n,m} + \Psi^k_{n,m-1}}{\Delta_2^2} = -\Omega^k_{n,m}, \tag{24}$$

$$\frac{P^k_{n+1,m} - 2P^k_{n,m} + P^k_{n-1,m}}{\Delta_1^2} + \frac{P^k_{n,m+1} - 2P^k_{n,m} + P^k_{n,m-1}}{\Delta_2^2} = -\left[\left(\frac{\partial U_1}{\partial X_1}\right)^2 + 2\frac{\partial U_1}{\partial X_2}\frac{\partial U_2}{\partial X_1} + \left(\frac{\partial U_2}{\partial X_2}\right)^2\right]^k_{n,m}. \tag{25}$$

Expressions (24), (25) are resulted from the use of the five-point differencing scheme (see Fig. 2) in Eqs. (9), (10), and have the second order of accuracy.

Discrete forms of the boundary and initial conditions. Discrete forms of the boundary and initial conditions (5), (11)-(15) and (16) are as follows

$$\Psi^k_{n,m}\Big|_{X_1=-L_u,\, L_1+L_{12}+L_2+L_d} = \frac{3}{2}X_2\left(1 - \frac{4}{3}X_2^2\right), \quad \Psi^k_{n,m}\Big|_{S^+_{ch},\, S^+_j} = \frac{1}{2}, \quad \Psi^k_{n,m}\Big|_{S^-_{ch},\, S^-_j} = -\frac{1}{2},$$

$$(\Psi^k_{n,m+1} - \Psi^k_{n,m-1})_{S_{ch},\, S^{\mathrm{h}}_j} = 0, \quad (\Psi^k_{n+1,m} - \Psi^k_{n-1,m})_{S^{\mathrm{v}}_j} = 0,$$

$$\Omega^k_{n,m}\Big|_{X_1=-L_u,\, L_1+L_{12}+L_2+L_d} = 12X_2, \quad \Omega^k_{n,m}\Big|_{S_{ch},\, S^{\mathrm{h}}_j} = -\frac{\Psi^k_{n,m+1} - 2\Psi^k_{n,m} + \Psi^k_{n,m-1}}{\Delta_2^2}\Bigg|_{S_{ch},\, S^{\mathrm{h}}_j},$$

$$\Omega\Big|_{S^{\mathrm{v}}_j} = -\frac{\Psi^k_{n+1,m} - 2\Psi^k_{n,m} + \Psi^k_{n-1,m}}{\Delta_1^2}\Bigg|_{S^{\mathrm{v}}_j}, \quad \left(\frac{\partial P}{\partial n}\right)^k_{n,m}\Bigg|_{S_{ch},\, S_j} = 0, \quad j = 1, 2. \tag{26}$$

$$\left(\frac{\partial P}{\partial X_1}\right)^k_{n,m}\Bigg|_{S^{\mathrm{v}}_j} = \left[\frac{1}{\mathrm{Re}}\left(\frac{(U_1)^k_{n+1,m} - 2(U_1)^k_{n,m} + (U_1)^k_{n-1,m}}{\Delta_1^2} + \right.\right.$$

$$+\frac{(U_1)^k_{n,m+1}-2(U_1)^k_{n,m}+(U_1)^k_{n,m-1}}{\Delta_2^2}\Bigg)-\frac{(U_1)^{k+1}_{n,m}-(U_1)^k_{n,m}}{\Delta_T}\Bigg]_{S_j^{\mathrm{v}}},$$

$$\left(\frac{\partial P}{\partial X_2}\right)^k_{n,m}\Bigg|_{S_{ch},S_j^{\mathrm{h}}}=\Bigg[\frac{1}{\mathrm{Re}}\Bigg(\frac{(U_2)^k_{n+1,m}-2(U_2)^k_{n,m}+(U_2)^k_{n-1,m}}{\Delta_1^2}+$$

$$+\frac{(U_2)^k_{n,m+1}-2(U_2)^k_{n,m}+(U_2)^k_{n,m-1}}{\Delta_2^2}\Bigg)-\frac{(U_2)^{k+1}_{n,m}-(U_2)^k_{n,m}}{\Delta_T}\Bigg]_{S_{ch},S_j^{\mathrm{h}}},$$

$$\Psi^k_{n,m}\Big|_{k=0}=0,\quad \Omega^k_{n,m}\Big|_{k=0}=0,\quad P^k_{n,m}\Big|_{k=0}=0 \tag{27}$$

(in the conditions for the pressure in (26), it is necessary to use representations (19)). Expressions (26) and (27) allow one to compute the values of all the terms of Eqs. (19), (23)-(25) on the boundary of the computational domain and at the initial time instant in this domain, respectively.

Solution of Eqs. (19), (23)-(25). The preliminary study of Eqs. (23), (24) indicates that at first sight Eq. (23) is nonlinear (because its right part is the function of $\Psi\Omega$) and these equations are coupled. However, more detailed analysis of the right part of (23) shows that all its terms are computed at the previous time moment, $T=T_k$, whereas at the initial time instant they are established. It means that the right part of (23) is the known quantity, and hence, (23) is just a computational scheme (rather than an equation) to find $\Omega^{k+1}_{n,m}$. Also, (23) and (24) is not a system of coupled algebraic equations.

Once the vorticity at all the points of the grid (17), (18) can be computed on the basis of the scheme (23), it is possible to solve the system of linear algebraic equations (i. e., SLAE) (24), which now has the known right part.

Usually, SLAEs are solved either by direct or iterative methods. The first ones are applied to small SLAEs and give satisfactory results. However, in case of big SLAEs (especially those having the rarefied matrices) the direct methods require a huge amount of computational time, and their application is unreasonable here. The iterative methods, when applied to SLAEs of big dimensions, require significantly less computational time, save the rarefaction degree of their matrices (when the matrices are rarefied) and give good results [5].

Proceed from these, as well as from the dimension and the rarefaction degree of the matrix of system (24), we choose the successive over-relaxation iteration method [5] to solve the system. The corresponding computational scheme for the discussed SLAE is of a second order of accuracy and looks as

$$\Psi^{k+1}_{n,m}=(1-\gamma)\Psi^k_{n,m}+\frac{\gamma}{2(1+\beta^2)}(\Psi^k_{n+1,m}+\Psi^k_{n-1,m}+\beta^2\Psi^k_{n,m+1}+\beta^2\Psi^k_{n,m-1}+\Delta_1^2\Omega^k_{n,m}) \tag{28}$$

(here $1<\gamma\leqslant 2$ and $\beta=\Delta_1/\Delta_2$). Since all the terms on the right in (28) are known, the quantities $\Psi^{k+1}_{n,m}$ on the left can be easily obtained.

The obtained values of Ψ allow one to compute the velocity components in (19) and further use them in (25). After that the above-noted successive over-relaxation method is applied to (25), viz.

$$P_{n,m}^{k+1} = (1-\gamma)P_{n,m}^{k} + \frac{\gamma}{2(1+\beta^2)}(P_{n+1,m}^{k} + P_{n-1,m}^{k} + \beta^2 P_{n,m+1}^{k} + \beta^2 P_{n,m-1}^{k} + \Delta_1^2 S_{n,m}^{k}), \quad (29)$$

$$S_{n,m}^{k} = \left[\left(\frac{\partial U_1}{\partial X_1}\right)^2 + 2\frac{\partial U_1}{\partial X_2}\frac{\partial U_2}{\partial X_1} + \left(\frac{\partial U_2}{\partial X_2}\right)^2\right]_{n,m}^{k}.$$

One can see that all the terms on the right in (29) are known. This permits one to find the pressure $P_{n,m}^{k+1}$ on the left in (29).

Conclusions.

1. A numerical technique has been devised to solve a problem of the steady laminar fluid motion in a straight plane hard-walled duct with two local axisymmetric rectangular constrictions.

2. This technique uses the stream function, the vorticity and the pressure as the basic variables, has the second order of accuracy in the spatial and the first order of accuracy in the temporal coordinates, provides high stability of a solution, and needs significantly less computational time to obtain a result compared to appropriate techniques available in a scientific literature.

3. The technique consists in: a) introducing the stream function and the vorticity, and subsequent transiting from the non-dimensional governing relations for the fluid velocity and the pressure to the corresponding non-dimensional relations for the stream function, the vorticity and the pressure; b) deriving their discrete counterparts in the nodes of the chosen space-time computational grid; c) integrating the systems of linear algebraic equations obtained after making the discretization.

4. The discretization is based on applying appropriate differencing schemes to the terms of the equations for the basic variables. These are the two-point temporal onward difference for the unsteady term of the vorticity equation, as well as the two-point backward differences (for its convective term) and the five-point approximations (for its diffusive term and for the Poisson's equations for the stream function and the pressure) in the axial and cross-flow coordinates. As for the velocity components, the appropriate central differences are applied to discretize their expressions.

5. The above-mentioned systems of linear algebraic equations for the stream function and the pressure are integrated by the iterative successive over-relaxation method. Regarding the algebraic relation for the vorticity, it is a computational scheme to find this quantity on the basis of the known magnitudes computed at the previous instant of time.

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А. О. Борисюк, https://orcid.org/0000-0002-3878-3915

Інститут гідромеханіки НАН України, Київ
E-mail: aobor@ukr.net

ЧИСЕЛЬНИЙ МЕТОД РОЗВ'ЯЗУВАННЯ ЗАДАЧІ ПРО РУХ РІДИНИ У ПРЯМОМУ ПЛОСКОМУ ЖОРСТКОМУ КАНАЛІ З ДВОМА ОСЕСИМЕТРИЧНИМИ ПРЯМОКУТНИМИ ЗВУЖЕННЯМИ. АЛЬТЕРНАТИВНИЙ ПІДХІД

Розроблено чисельний метод розв'язування задачі про стаціонарний ламінарний рух рідини у прямому плоскому жорсткому каналі з двома локальними осесиметричними прямокутними звуженнями. У цьому методі як основні змінні використовуються функція течії, завихореність і тиск. Він має другий порядок точності по координатах і перший порядок точності по часу, забезпечує високу стійкість розв'язку і потребує значно менше комп'ютерного часу для одержання результату порівняно з відповідними методами, описаними в науковій літературі. За цим методом сформульована задача розв'язується шляхом: а) введення функції течії і завихореності та подальшого переходу від безрозмірних співвідношень для швидкості і тиску до відповідних безрозмірних співвідношень для функції течії, завихореності і тиску; б) виведення дискретних аналогів цих співвідношень у вузлах вибраної просторово-часової решітки; в) інтегрування систем лінійних алгебраїчних рівнянь, одержаних внаслідок проведення зазначеної дискретизації. Дискретизація ґрунтується на застосуванні відповідних різницевих схем до членів рівнянь для введених змінних. Це — одностороння різниця вперед для нестаціонарного члена рівняння переносу завихореності, а також односторонні різниці проти потоку (для конвективного члена цього рівняння) та п'ятиточкові шаблони (для його дифузійного члена та рівнянь Пуассона для функції течії і тиску) по осьовій та поперечній координатах. Що стосується компонент швидкості, то для дискретизації їхніх виразів застосовуються відповідні центральні різниці. Зазначені вище системи лінійних алгебраїчних рівнянь для функції течії і тиску інтегруються ітераційним методом послідовної верхньої релаксації. Алгебраїчне ж співвідношення для завихореності не потребує застосування ніякого методу розв'язування, оскільки вже є розрахунковою схемою для визначення цієї величини на основі відомих величин, знайдених у попередній момент часу.

Ключові слова: *рух рідини, плоский канал, прямокутне звуження, метод.*