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INERTIAL NAVIGATION ACCURACY AND ACCELEROMETERS WITH DAMPING

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Abstract**—***Nowadays an anxiously important problem is the creation of the low-cost gyro-free inertial navigation systems. That is why we focus primarily on linear accelerometer to avoid gyro using. The problem of inertial navigation accuracy with using of accelerometers in inertial measurement unit are discussed. It is shown that a strict successive analysis of accelerometer measurement procedure makes it possible to understand probable new source of errors in the value estimation of the disturbing force magnitude. If do not take into account these new possible errors any self-adjusting system of an airplane may be functionally unreliable. These errors arise primarily when temporal duration of disturbing force is less than some critical time τ, which depends on the given characteristics of an aircraft Inertial Navigation System and aircraft itself. The method of finding τ is proposed. A presence of a damping force in the proof mass motion equation is examined.*

Index Terms—Inertial navigation system; accelerometer; wave equation; disturbing influences Lagrange function; inertial management unit.

I. INTRODUCTION

Committee of ICAO on future navigational systems (FANS – Future Air Navigation System) made decision about the obligatory use of satellite navigation in combination from inertial navigation systems (INS) [1]. Nowadays many researchers all over the world are concentrating their efforts in order to create gyro-free INS based on the usage of accelerometers only [2]. There are at least two reasons of this activity. The 1st reason is a growth of the MEMS – technology, producing low-cost sensors with enhanced accuracy characteristics, thus facilitating creation of ABSINS. The 2nd reason is the emerging of some difficulties of attitude determination based on gyros application in the case of navigation and control of the rapidly rotating moving vehicle (MV) and the large accelerations values.

The problems of navigation accuracy whose roots lay back in the mid-twentieth century should be extended now to meet the new challenges. Amongst these well-known challenges it is necessary to mention enormously high frequency of flights, continuously disproving environmental conditions, problems with airport landing. In the first position the terrorist incidents in civil aviation should be placed which include aircraft hijacking, airlines bombing, terrorist attacks on airports. Common future of all above mentioned items is the extremely short segment of time available for checking out the health of the avionic system on the flight line or on the ground. Namely, this common point makes it possible to gather consideration of all these distinct situations within one theoretical framework and elaborate mathematical tools to construct admissible diagnostic procedures [3] – [5].

II. ACCELEROMETER PRINCIPLE

Knowing that the main aim is to focus primarily on linear accelerators, let us very shortly recollect the main results of previously considerations. The very imported part of INS is so called Inertial Measurement Unit (IMU), which consists of accelerometers and gyros. Let us restrict to exam accelerometer only. The very short review of results of article [5] may be present as follows.

In the any local inertial frame (LIF) the total disturbing external force F^{ext} according to the center-of-mass theorem gives birth to the acceleration a_c of airplane

$$a_c = \frac{F^{\text{ext}}}{M}, \tag{1}$$

where, M is airplane instant total mass which includes masses of all internal devices and instant amount of fuel. Of course, it is trivial approximate result ignoring after all the continuous descent of fuel. A strict successive analysis of accelerometer measurement procedure makes it possible to derive the next exact motion equation for accelerometer proof mass m. according to some selected direction

$$F(t) = -k_1 \Delta l_1(t) + k_2 \Delta l_2(t) - m a_c, \tag{2}$$

where $k_1, \Delta l_1(t)$, stand accordingly for elasticity factor and a change in length of right spring, index

"2" means the left spring. For simplicity let us set $k_1 = k_2 = 2k$, so that k means the elasticity of whole spring as a result of the consecutive spring connection. Under condition $l_1 + l_2 = l$ = const have $dl_1 = - dl_2$, so $\Delta l_1 = -\Delta l_2$; $|\Delta l_1| = |\Delta l_2| = \frac{\Delta l}{2}$, Substitution of these relation and as well as (1) into (2), in the case when $F(t) = 0$ gives

$$\Delta l = \frac{mF^{\text{ext}}}{2Mk} . \qquad (3)$$

Condition $F(t) = 0$ means that Δl corresponds to the proof mass equilibrium position.

But absence of forces means absence of accelerations but not of velocities. How accelerometers proof mass approaches to its halt? How its position depends on time? The answers on these questions may describe correlation between duration of influence and data for self-adjusting system and somehow avoid extremely high errors in the estimation of the disturbing force. Purposely temporarily ignoring damping to make clear some principal points and using the work-energy theorem, also known as kinetic energy increase theorem, for the change in the kinetic energy ΔT it is obtained

$$\Delta T = \int_0^{\Delta l(\tau)} (2k\Delta l(t) - ma_c)dx. \qquad (4)$$

Suppose that accelerometer mass approaches to its halt at time $\Delta t = \tau$, which means $\Delta T = 0$, so (4) may be rewritten as

$$2\int_0^{\Delta l(\tau)} k\Delta l(t)dx - m\int_0^{\Delta l(\tau)} ma_c dx = 0 . \qquad (5)$$

After integration taking into account that $\Delta l(t) = x$ and integration using all above mentioned relations the simple equation is written

$$k(\Delta l(\tau))^2 - ma_c \Delta l(\tau) = 0 . \qquad (6)$$

Equation (6) has two solutions: first $\Delta l(\tau) = 0$ that means the initial position, when disturbing forces begin to act and second

$$\Delta l(\tau) = \frac{ma_c}{k} = \frac{mF^{\text{ext}}}{Mk}, \qquad (7)$$

from which it is possible to derive relation which defines position of an accelerometer proof mass, where it approaches to its halt, noting that $\Delta l(\tau) = 2\Delta l$, where k stands for elasticity factor of proof mass spring and Δl corresponds to its equilibrium position. It's clear that the accelerometer proof mass vibrates between these two positions. Obviously, it's necessary to introduce damping which approaches proof mass to it's equilibrium position. Suppose now that time segment Δt of disturbing force F^{ext} action is less than $\Delta t < \tau$, i.e. In this case using in all existing IMU schemes relation (3) is incorrect and there are enormously large errors in evaluating of F^{ext}. So, let us should return to the equation (4) in which at this time $\Delta T \neq 0$ and have opportunity to find answer the questions what should be after the moment of a disappearance of a disturbing force F^{ext} and how it is possible to properly evaluate the F^{ext} in this case. After the moment when external influence become extinct accelerometer proof mass m continues to move. For this case methods of estimation of proper time duration are elaborated in [5], where by means of application of Lagrange function was found that Δt may be evaluated as

$$\Delta t = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - U(x) - ma_c}} + \tau_0 . \qquad (8)$$

where, $U = \frac{mx^2}{2}$, and $E = \frac{m\dot{x}^2}{2} + ma_c x + U$.

Lagrange function L in accelerated frame has the form [7]:

$$L = \frac{m\dot{x}^2}{2} - ma_c x - U , \qquad (9)$$

E and τ_0 are integration constants they play role of fitting parameters, which allow apply this quite general framework to every given real data.

III. Accelerometer with Damping

The previous consideration shows the extremely important role of damping forces in the realization of any practically significant scheme to construct an accelerometer. To study in details these problems let us recollect widely known facts of oscillation motion theory. The considered case differs from classical theory of oscillation only in the selection of equilibrium position. To the strict analysis of this very impotent point will be devoted future publications. The equation when damping forces are present has the form [6]

$$m\ddot{x} = -kx - r\dot{x} , \qquad (10)$$

where, m denotes accelerometer proof mass; k stands for elasticity factor; r is a constant called the resistance coefficient. This equation can be rewritten as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \qquad (11)$$

where $2\beta = \frac{r}{m}$, $\omega_0^2 = \frac{k}{m}$.

Introduction of the function $x = \exp(\lambda t)$ into (11) leads to the characteristic equation

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0. \quad (12)$$

The roots of this equation are

$$\lambda_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}, \quad (13)$$

$$\lambda_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}. \quad (14)$$

When the damping is not too great (at $\beta < \omega_0$) the radicand will be negative. Let us write it in the form $(i\omega)^2$, where ω is a real quantity equal to

$$\omega = \sqrt{\omega_0^2 - \beta^2}. \quad (15)$$

Here, the roots of the characteristic equation will be as follows:

$$\lambda_1 = -\beta + i\omega, \quad \lambda_2 = -\beta - i\omega. \quad (16)$$

The general solution of (6) will be the function

$$x = C_1 e^{(-\beta+i\omega)t} + C_2 e^{(-\beta - i\omega)t} = e^{-\beta t}(C_1 e^{i\omega t} + C_2 e^{-i\omega t}) \quad (17)$$

or

$$x = A_0 e^{-\beta t}\cos(\omega t + \alpha), \quad (18)$$

where α can be determined by means of expressions

$$C_1 = \frac{A}{2}e^{i\alpha}, \quad C_2 = \frac{A}{2}e^{-i\alpha}. \quad (19)$$

The graph of (18) is shown below (Fig. 1).

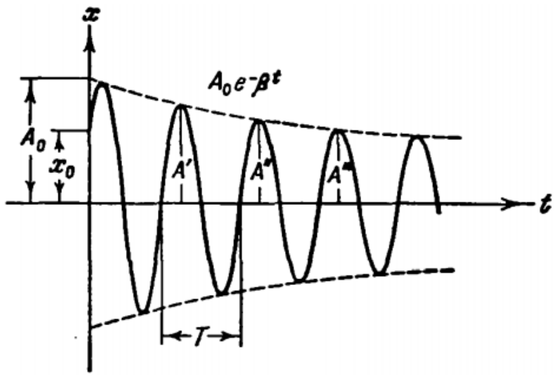

Fig. 1. Oscillation with damping

IV. The Rate of Damping

The rate of damping of oscillations is determined by the quantity $\beta = r/2\,e$ defined as the damping factor. Let us find the time τ during which the amplitude diminishes e times. By definition, $e^{-\beta\tau} = e^{-1}$, whence $\beta\tau = 1$. Consequently, the damping factor is the reciprocal of the time interval during which the amplitude diminishes e times. In analogy with

$$T_0 = \frac{2\pi}{\omega_0}. \quad (20)$$

The period of damped oscillations is

$$T = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}. \quad (21)$$

When the resistance of the medium is insignificant, the period of oscillations virtually equals $T_0 = 2\pi/\omega_0$. The period of oscillations grows with an increasing damping factor.

The following maximum displacements to either side (for example A', A'', A''', etc.) form a geometrical progression. Indeed, if $A' = A_0 \exp(-\beta t)$, then $A'' = A_0 \exp[-\beta(t + T)] = A' \exp(-\beta T)$, $A''' = A_0 \exp[-\beta(t+2T)] = A'' \exp(-\beta T)$, etc.

In general, the ratio of the values of the amplitudes corresponding to moments of time differing by a period is

$$\frac{A(t)}{A(t+T)} = e^{\beta T}. \quad (22)$$

This ratio is called the damping decrement, and its logarithm is called the logarithmic decrement:

$$\Lambda = \ln\frac{A(t)}{A(t+T)} = \beta T. \quad (23)$$

To characterize an oscillatory system, the logarithmic decrement Λ is usually used. Expressing β through Λ and T in accordance with (14), the law of diminishing of the amplitude with time can be written in the form

$$A = A_0 \exp\left(-\frac{\Lambda}{T}t\right). \quad (24)$$

In the interval during which the amplitude diminishes e times, the system manages to complete $N_e = \tau/T$ oscillations. Let us find from the condition $\exp(-\lambda\tau/T) = \exp(-1)$ that $\lambda\tau/T = \lambda N_e = 1$. Hence the logarithmic decrement is the reciprocal of the number of oscillations completed during the interval in which the amplitude diminishes e times An oscillatory system is often also characterized by the quantity

$$Q = \frac{\pi}{\Lambda} = \pi N_e, \quad (25)$$

called the quality, or simply the Q, of the system. As can be seen from its definition, the quality is proportional to the number of oscillations N_e performed by the system in the interval τ during which the amplitude of the oscillations diminishes e times.

Accordingly, the energy of the system in damped oscillations diminishes with time according to the law

$$E = E_0 e^{-2\beta t}, \tag{26}$$

(E_0 is the value of the energy at $t = 0$). Time differentiation of this expression gives the rate of growth of the system's energy:

$$\frac{dE}{dt} = -2\beta E_0 e^{-2\beta t} = -2\beta E. \tag{27}$$

By reversing the signs, is found the rate of diminishing of the energy:

$$-\frac{dE}{dt} = 2\beta E \ . \tag{28}$$

If the energy changes only slightly during the time equal to a period of oscillations, the reduction of the energy during a period can be found by multiplying (28) by T:

$$-\Delta E = 2\beta TE, \tag{29}$$

(ΔE stands for the increment, and $-\ \Delta E$ for the decrement of the energy). Finally at the relation is obtained

$$\frac{E}{(-\Delta E)} = \frac{Q}{2\pi}, \tag{30}$$

from which it follows that upon slight damping of oscillations, the quality with an accuracy up to the factor 2π equals the ratio of the energy stored in the system at a given moment to the decrement of this energy during one period of oscillations.

It follows from (21) that a growth in the damping (Fig. 2) factor is attended by an increase in the period of oscillations.

At $\beta = \omega_0$ the period of oscillations becomes infinite, i.e. the motion stops being periodic. At, $\beta > \omega_0$ the roots of the characteristic equation become real, and the solution of the differential equation is equal to the sum of two exponents:

$$x = C_1 e^{-\lambda_1 t} + C_2 e^{-\lambda_2 t}. \tag{31}$$

Here C_1 and C_2 are real constants whose values depend on the initial conditions:

$$[\text{on } x_0 \text{ and } v_0 = (\dot{x})_0].$$

The motion is therefore aperiodic – a system displaced from its equilibrium position returns to it without performing oscillations. Previous graph shows two possible ways for a system to return to its equilibrium position in aperiodic motion. How the system arrives at its equilibrium position depends on the initial conditions.

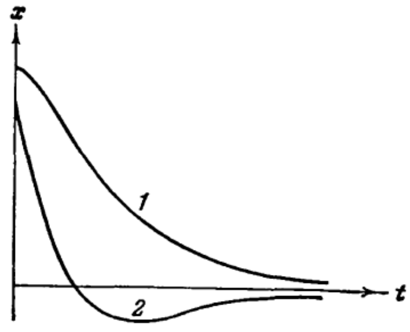

Fig. 2. Aperiodic motion

The motion depicted by curve *2* (Fig.2) is obtained when the system begins to move from the position characterized by the displacement x_0 to its equilibrium position with the initial velocity v_0 determined by the condition:

$$|v_0| > |x_0|\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right). \tag{32}$$

This condition will be obeyed when a system brought out of its equilibrium position is given a sufficiently strong impetus toward it. If after displacing a system from its equilibrium position it is released it without an impetus (i.e. with $v_0 = 0$) or impart to it an impetus of insufficient force such that v_0 is less than the value determined by the condition (32), the motion will occur according to curve *1* in graph (see Fig. 2).

V. Conclusions

A strict successive analysis of accelerometer measurement procedure makes it possible to understand probable new sources of errors in the value estimation of the disturbing force magnitude. These sources include also the character of the rate of damping. A growth in the damping factor is attended by an increase in the period of oscillations. At $\beta = \omega_0$ the period of oscillations becomes infinite, i.e. the motion stops being periodic. At, $\beta > \omega_0$ the roots of the characteristic equation become real. The motion is therefore aperiodic – a system displaced from its equilibrium position returns to it without performing oscillations. How the system arrives at its equilibrium position depends on the initial conditions and all these circumstances are the probable new sources of additional errors. The consequences of such errors may be anxiously substantial. If don't take into account these new possible errors any self-adjusting system of an airplane may be functionally damaged or become almost unreliable. These errors arise primarily when temporal duration of disturbing influence is less than some critical time τ, which depends on some characteristics of an aircraft INS, and aircraft itself.

The method of finding τ is elaborated in [5]. Further descend of disturbing duration makes all above discussed INS methods to be extinct, and influences on aircrafts should be described by means of 3D wave equation and its solutions, known as plane waves. For the investigation of these problems and also for the consideration of the extended (to involve damping) previous analysis will be devoted our prospective publications.

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О. А. Іванов, Ю. А. Опанасюк Точність інерційної системи навігації та акселерометри з демпфуванням

На сьогодні дуже важливою проблемою є створення низьковартістних інерційних навігаційних систем що базуються на використанні лише акселерометрів. Обговорюються питання точності таких навігаційних систем. Показано, що послідовний прецизійний аналіз функціонування акселерометра вказує на нові можливі джерела похибок при оцінювані величини збурюючого впливу. Неврахування таких факторів призводить до функціональної ненадійності системи навігації. Розглянуто ефект наявності коефіцієнту загасання у рівнянні руху пробної маси акселерометра.

Ключові слова: інерційна система навігації; хвильове рівняння; збурюючі впливи.

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А. А. Иванов Ю. А. Опанасюк. Точность инерциальной системы навигации и акселерометры с демпфированием

В настоящее время важной проблемой является создание недорогих инерционных навигационных систем основанных на использовании только акселерометров. Обсуждаются вопросы точности таких навигационных систем. Показано, что последовательный прецизионный анализ функционирования акселерометра позволяет обнаружить новые возможные источники погрешностей в оценке величины возмущающего воздействия. Игнорирование этих факторов может повлиять на функциональную надежность системы навигации. Рассмотрен эффект наличия коэффициента затухания в уравнении движения пробной массы акселерометра.

Ключевые слова: инерционная система навигации; волновое уравнение; возмущающие воздействия.

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