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A NOTE ON THE GRAPHICAL REPRESENTATION OF THE DERIVATIVES

Abstract. *In the article at hands an alternative definition of the concept of the derivative is presented, which makes no use of limits. This definition is based on an old idea of Descartes for calculating the slope of the tangent at a point of a curve and holds for all the algebraic functions. Caratheodory extended this definition to a general definition of the derivative in terms of the concept of continuity. However, although this definition has been used successfully by many German mathematicians, it is not widely known in the international literature, nor it is used in the school book texts. After presenting Caratheodory's definition, the article closes by describing methods for calculating the derivative at a point of a function y = f(x) with the help of a suitably chosen table of values of f(x), and for designing of the graph of the derivative function f΄(x) given the graph, but not the formula, of f(x). These methods are based on the graphical representation of the derivative, which should be reclaimed better in general for teaching purposes.*

Key words: *teaching/learning the derivatives, tangent at a point of the graph of a function, algebraic and transcendental functions, Caratheodory's definition of the derivative, graphs of the derivative functions, Maple software.*

Problem formulation. Mathematical education researchers report that, although student routine performance on differentiation items is usually adequate, most of them have little intuitive or conceptual understanding of the derivative concept [1-7], etc. Many calculus students are, for example, proficient at differentiating a function and finding its critical values. However, can students conceptualize these actions and work with them if they are not presented in equation form? For example, Ferrini-Mundy & Graham [3] described student difficulties in trying to sketch the derivative of a given function presented only graphically. Many of them tried first to find an algebraic function's representation.

It seems that many students perform poorly due to their weakness to deal successfully with information about abstract concepts (e.g. functions, limits etc.) given in symbolic form, but also because they luck of the necessary cognitive schemas that could allow them to organize their knowledge on those matters [8]. Consequently, the use by the instructor of various representations of the concept of the derivative is recommended as well as the student orientation to analogous activities [9-12], etc. This is reinforced by Tall [13] who has made a strong argument for direct links between visualization and symbolization when teaching the concept of the derivative. In particular, many researchers focus their attention on the *graphical representation* of the derivative [3, 14, 15], etc.

On the other hand, a number of authors have used principles of the *APOS (action-process-object-schema)* theory and of the connected to it *ACE (actions on computers, classroom discussion, homework exercises)* teaching cycle for the research on the improvement of the understanding of the concept of the derivative [5. 6, 16, 17], etc. The APOS theory, which is based on Piaget's principles for learning [18], was developed in the USA during the 1990s by Ed Dubinsky and his collaborators and has been used for improving the teaching and understanding of almost all mathematical topics [16, 19-22], etc.

Giraldo, Carvalho & Tall [23] distinguish between the ***description*** of a concept, which specifies the main properties of that concept and the formal concept's ***definition***. In particular, they note that a commonly used description of the derivative of a function $y = (x)$ at a point a of its domain is that it gives the ***slope of the tangent line*** to the function's graph at the point $(a, f(a))$. It is also recalled that another common description of the derivative is that it expresses the ***rate of change*** of the function $f(x)$ with respect to x, while its ***physical meaning*** is connected to the speed and to the acceleration at a moment of time of a moving object under the action of a steady force. On the contrary, the formal definition of the derivative $f'(a)$ of $y = f(x)$ at a is given by

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

However, the concept of the limit is difficult to be understood by students. For example, in calculating the limit appearing in definition (1), while we initially accept that x tends to α with values $x \neq a$, we finally set $x = a$. Although this manipulation can be easily justified by the continuity of $y = f(x)$ at α, it is frequently confusing the novices. It is characteristic that 150 whole years were required from the time that Newton (1642-1727) and Leibniz (1646-1716) initiated the Differential Calculus until the complete understanding of the concept of the limit and this says everything!

Therefore, it is natural for one to put the question of whether or not is it possible to initially define the concept of the derivative without using the limits. In fact, as we shall see next, this is possible for the wide class of ***algebraic functions.*** Nevertheless, although this is based to an old idea of Rene Descartes (1596-1650), the corresponding definition of the derivative is not so much in use in our days, nor it is mentioned in the school texts.

The purpose of the present article is to describe this method that results to an algebraic definition of the derivative. Nevertheless, one cannot use this definition in case of the ***transcendental functions*** (trigonometric, exponential, logarithmic and many others), where the use of the traditional definition (1) becomes necessary. After all, this is actually the main difference between Algebra, where only finite processes are in use, and Mathematical Analysis, where the use of infinite processes, including the limits, becomes necessary.

The rest of this article is organized as follows: In the second Section the algebraic method of Descartes is presented for calculating the slope of the tangent at a point of the graph of an algebraic function and the resulting algebraic definition of the derivative is given. As an example, the product and the chain rule for the derivatives are proved with the help of this definition. In third Section a ***general*** definition for the derivative in terms of the concept of continuity, due to Caratheodory, is given, which is based on the previous definition for the algebraic functions. Finally, in fourth Section approximate methods of calculating the derivative at a point of a function and of designing the graph of $f'(x)$ when the graph, but not the formula, of the function $y = f(x)$ is given, are presented and our final conclusion is drawn.

The method of Descartes for calculating the slope of the tangent at a point of an algebraic curve. Descartes [24, 25] worked as follows for calculating the slope of the tangent at a point P (α, α^2) of the ***parabola*** $y = f(x) = x^2$: The equation of a secant line of the parabola through P is of the form

$$y - \alpha^2 = \lambda(x - \alpha), \tag{2}$$

where λ is its slope. Equation (2) can be also written as

$$x^2 - \alpha^2 = \lambda(x - \alpha),$$

or as

$$(x + \alpha)(x - \alpha) - \lambda(x - \alpha) = 0,$$

or finally as

$$(x - \alpha)(x + \alpha - \lambda) = 0.$$

The above secant line becomes a tangent of the parabola at P if, and only if, ***the multiplicity of the root $x = \alpha$ is greater than 1.*** This means that the value $x = \alpha$ need to be also a root of the equation

$$x + \alpha - \lambda = 0,$$

which gives that $\lambda = 2\alpha$. Consequently, in terms of the geometric representation of the derivative, which was developed later by Newton and Leibniz, one finds that for $f(x) = x^2$ we have $f'(\alpha) = 2\alpha$.

It must be mentioned of course that in the era of Descartes the equation (2) of a straight line in terms of its slope was not known. This means that, in order to determine the equation of a secant line of the parabola through P, the consideration of another point of the parabola, apart from P, was needed, which makes the above procedure slightly more complicated. However, the brilliant idea of Descartes, who was a deep expert in Algebra, remains unchanged.

Generalizing the above process let as now consider a ***polynomial*** of any degree n with real coefficients of the form

$$y = f(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots\ldots + b_1 x + b_0.$$

For every real number α we have that

$$f(x)-f(\alpha)=b_n(x^n-\alpha^n)+b_{n-1}(x^{n-1}-\alpha^{n-1})+\ldots+b_1(x-\alpha)=b_n(x-\alpha)(x^{n-1}+x^{n-2}\alpha + \ldots. + \alpha^{n-1}) + b_{n-1}(x-\alpha)(x^{n-2} + x^{n-3}\alpha + \ldots. +\alpha^{n-2}) +\ldots+ b_1(x-\alpha),$$

or

$$f(x) - f(\alpha) = q(x)(x - \alpha), \tag{3}$$

where $q(x)$ is also a polynomial with real coefficients and degree less than n.

Then, like in parabola, each secant line of the graph of the polynomial function $y = f(x)$ through the point P

($\alpha, f(\alpha)$) with slope equal to λ, it has an equation of the form

$$f(x) - f(\alpha) = \lambda(x - \alpha),$$

which using (3) could be written as

$$(x - \alpha)[q(x) - \lambda] = 0.$$

Therefore, applying again Descartes's argument for the parabola, one obtains that the slope of the tangent of $y = f(x)$ at a is equal to $\lambda = q(\alpha)$, which shows that $f\,'(\alpha) = q(\alpha)$.

Note that equation (3) holds even in case of a straight line

$$f(x) = b_1x + b_0$$

with $b_1 \neq 0$ or $b_1 = 0$, where

$$f(x) - f(\alpha) = b_1 (x - \alpha) \text{ and } \lambda = b_1.$$

In other words, another straight line intersects the line

$$y = b_1x + b_0$$

with multiplicity greater than 1, if, and only if, it coincides with it.

Now, let us consider a ***rational function*** of the form

$$F(x) = \frac{f(x)}{g(x)},$$

where $g(x)$ και $f(x)$ are polynomials, with $g(x) \neq 0$. Then, by (3)

$$f(x) - f(\alpha) = q(x)\,(x - \alpha).$$

Therefore $F(x) = \frac{f(x)}{g(x)}\,(x - a) = Q(x)(x - a)$, where $Q(x) = \frac{q(x)}{g(x)}$ is a rational function. Therefore, we find again that

$$F(x) - F(\alpha) = Q(x)(x - a).$$

Consequently equation (3) also holds in this case , with $Q(x)$ a rational function having the same domain with $F(x)$.

Further, let us consider the function $f(x) = \sqrt{x}$. Then, if $\alpha>0$, one can write

$$f(x) - f(\alpha) = \sqrt{x} - \sqrt{a} = \frac{1}{\sqrt{x}+\sqrt{a}}\,(x - a).$$

Therefore, equation (3) holds again with

$$q(x) = \frac{1}{\sqrt{x}+\sqrt{a}}.$$

It is recalled here that a function is said to be an ***algebraic function***, if it is a root, with respect to y, of an equation of the form

$$f_n(x)y^n + f_{n-1}(x)y^{n-1} + \ldots\ldots + f_1(x)y + f_0(x) = 0,$$

where n is a natural number and $f_n(x), f_{n-1}(x), \ldots, f_1(x), f_0(x)$ are polynomials, with respect to x, with real coefficients.

From the above definition it becomes evident that all the rational and ***irrational functions*** (i.e. those containing roots of rational functions of any order) are algebraic functions.

By suitably generalizing the above used arguments for rational functions, one can prove [26] that equation (3) remains true for all algebraic functions. Namely we have the following ***factorization lemma***.

Lemma. For each a in the domain of an algebraic function $y = f(x)$ there exists an algebraic function $q(x)$ having the same domain with $f(x)$, such that

$$f(x) - f(\alpha) = q(x)\,(x - \alpha).$$

As an immediate consequence of the above Lemma, one can give the following algebraic definition for the derivate of an algebraic function.

Definition.

I) The ***derivative number*** at a point a in the domain of an algebraic function $y = f(x)$ is defined by

$$f\,'(\alpha) = q(a).$$

II) The ***derivative function*** $y = f\,'(x)$ is defined in a suitable subset of the domain of $f(x)$ by

$$f\,'(x) = q(x).$$

Example. The domain of $f(x) = \sqrt{x}$ and of $q(x)= \frac{1}{\sqrt{x}+\sqrt{a}}$ is the interval $[0, +\infty)$, while the domain of $f'(x)= \frac{1}{2\sqrt{x}}$ is restricted to $(0, +\infty)$. When $a = 0$, then $q(x)= \frac{1}{\sqrt{x}}$ and equation (3) gives that $\sqrt{x} = \frac{x}{\sqrt{x}}$. Also, for $x = 0$, $q\,(0) = \frac{1}{\sqrt{a}}$ and equation (3) is trivially written as $-\sqrt{a} = \frac{-a}{\sqrt{a}}$.

Remarks. 1) Since all algebraic functions are ***continuous*** functions, it becomes evident that the above algebraic definition of the derivative is equivalent to the classical definition (1). In fact the factorization described by equation (3) actually raises the vagueness $\frac{0}{0}$ appearing in the limit of definition (1).

However, if a ***transcendental function*** *y = f(x)* is not continuous at *α,* then the above limit either does not exist or it becomes infinite. Consequently, in such cases the derivative *f′(a)* does not exist, which means that the algebraic definition of the derivative cannot be applied in this case.

2) Note that all the known properties of the derivatives can be proved with the help of the above algebraic definition. As a first example, we shall prove here the ***product rule*** for derivatives. For this, let us consider the algebraic functions *y = f(x)* and *y = g(x)* with common domain A. Then, for each *α* in A, we can write

$$f(x) - f(\alpha) = q_1(x)(x - a)$$

and

$$g(x) - g(\alpha) = q_2(x)(x - a),$$

where $q_1(x)$ and $q_2(x)$ are also algebraic functions with domain A. Then

$$f'(\alpha) = q_1(\alpha) \text{ and } g'(\alpha) = q_2(\alpha).$$

But $fg(x) - fg(a) = f(x)g(x) - f(\alpha)g(\alpha) = f(x)g(x) - f(\alpha)g(a) + f(x)g(a) - f(x)g(a) = f(x)[g(x) - g(a)] + [f(x) - f(a)]g(a) =$

$$= f(x)\, q_2(x)(x-a) + g(a)\, q_1(x)(x-a) = [f(x)q_2(x) + g(a)q_1(x)](x-a) = q(x)(x-a).$$

Therefore,

$$(fg)'(\alpha) = q(\alpha) = f(\alpha)q_2(\alpha) + g(a)q_1(\alpha) = f(\alpha)g'(\alpha) + g(a)f'(\alpha) = (fg' + gf')(\alpha).$$

3) The proof of the ***chain rule*** is even easier. In fact, if *F = f o g*, with *y = g(x),* then applying equation (3) one finds that

$$F(x) - F(a) = q(x)(x - a) \tag{4}$$

and $g(x) - g(a) = q_2(x)(x - a)$.

Also, for *b=g(a),* one finds with respect to *y* that

$$f(y) - f(b) = q_1(y)(y - b).$$

But $F(x) - F(a) = f(y) - f(b) = f(g(x)) - f(g(a)) = q_1(g(x))[g(x) - g(a)] = q_1(g(x))q_2(x)(x - a).$ (5)

On comparing equations (4) and (5) one obtains that $q(x) = q_1(g(x))q_2(x)$.

Therefore, $q(a) = q_1(g(a))q_2(a)$ or $q(a) = q_1(b)q_2(a)$.

Thus, applying the algebraic definition of the derivative one finally finds that

$$(f \circ g)' = f_x{}' = f_y{}'\, y_x{}'.$$

4) The common experience shows that, after proving the properties and finding the basic formulas for calculating the derivatives, students are very little interested for using the limits connected to them. In fact, their attention is then turned to the study of functions and to the design of their graphs with the help of derivatives, as well as to the other practical applications of the derivatives, which are also useful to the study of other scientific topics, like Physics, Economy, Engineering, etc. Therefore, one could be taught the derivatives on the basis of their algebraic definition first with the introduction of the difficult concept of the limits to be followed only when it becomes necessary, i.e. when it is time to calculate the derivatives of the transcendental functions [27].

Caratheodory's definition for the derivative. Let as consider a polynomial function *y = f(x)* in a real interval δ of radius *α*. Since f(x) is a bounded function in δ, equation (3) gives that

$$|f(x) - f(\alpha)| \leq K|x-\alpha|, \tag{6}$$

for a suitable value of the constant K. Therefore, as the interval δ decreases around *α*, the value of *f(x)* approximates the value *f(a)*.

In this simple way one can introduce the concept of ***continuity*** of a function and can prove that all polynomials are continuous functions, without making a direct use of the limits. Note that inequality (6) holds for all algebraic functions [26], which means that one can prove through it the continuity of all such functions.

The famous Greek mathematician Caratheodory (1873-1950), who, among others, helped Einstein to develop the mathematical part of the Relativity Theory, introduced the following ***general*** definition for the derivative in terms of the concept of continuity [28], which extends the definition for the algebraic functions presented in Section II.

Definition. A function *y = f(x) is* said to be ***differentiable*** at a point *x = a* in its domain, if there exists a factorization of the form

$$f(x) - f(\alpha) = q(x)\,(x - \alpha),$$

where *q(x)* is a function ***continuous*** at *α.* Then the value *q(α)* is called the ***derivative*** (number) of *y = f(x)* at *α* and it is denoted by *f ′(α).* Further, the function *y = f ′(x)*, defined in a suitable subset of the domain of *f(x)*, is called the derivative (function) of *f(x).*

Since in the above definition *q(x)* is continuous at *a* , we have that

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{q(x)(x-a)}{x-a} = q(\alpha).$$

Therefore, Caratheodory's definition is equivalent to the classical definition (1) of the derivative.

Although the above definition has been used successfully by a considerable number of German mathematicians of the era of Caratheodory and later on, it is not so widely known in the international and in particular in the English literature, neither it is included in the school text books.

Of course, the above definition is not completely independent from the concept of the limit, which is strictly connected to the concept of continuity. Nevertheless, it has the advantage that at the first steps (algebraic functions) it does not require the use of the limits at all. Consequently, it could be reclaimed more for teaching purposes.

Approximate calculation of the derivative and design of its graph. The examples to be presented in this Section constitute part of a joint, but unpublished yet, work of the present author with V. Borji and S.H. Alamolhodaee from Ferdowsi University of Mashhad, Iran, on the use of the APOS theory for the graphical representation of the derivative.

The geometric representation of the derivative of a function at a point of its domain leads to a method of calculating the derivative, if we don't know its formula, with the help of a Table of suitably chosen values of the given function. This method, which is applied in Example 1, is based on the fact that the secant line of the graph of a function *y = f(x)* at the points A and B approximates the tangent of the graph at A, when the point B is moving approaching to A either from the left, or from the right (see Figure 1, where B is approaching A from its right side).

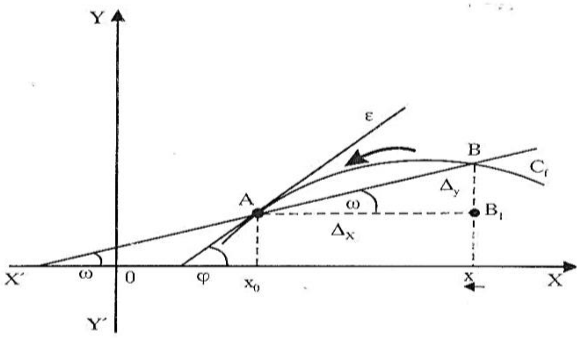

Fig. 1. The point B approaches A from its right side

Example 1. Use the values of the continuous function *y = f(x)* given in the below Table to approximate the value of its derivative at *x* = 2.

Table1. Values of y = f(x) in Example 1

x	1.9	1.99	1.999	…	2.0001	2.001	2.1
F(x)	3.61	3.9601	3.996001	…	4.00040001	4.004001	4.41

From Table 1 one observes that, as *x* tends to 2, *f(x)* tends to 4. Then the slopes of the corresponding secant lines of the function's graph approaching its tangent at (2,4) can be calculated. This is shown in Fig.2, concerning a correct student solution of this example.

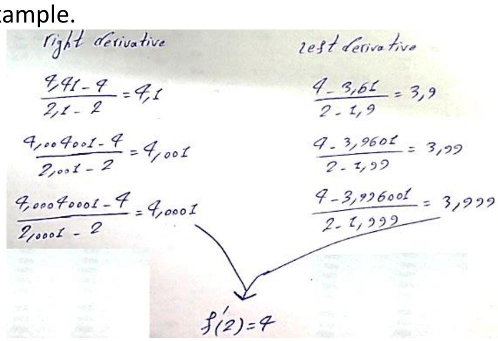

Fig.2 A correct student solution of Example 1

In this way, it is finally found that that *f* ′(2) = 4.

The method applied in Example 1, combined with the use of the proper software, can be also utilized for the approximate design of the graph of the function *f* ′(*x*), given the graph, but not the formula, of *f(x).* For this, it suffices to calculate graphically the coordinates (*x, f(x)*) of *n* random points of the graph of *f{x)*, where *n* is a sufficiently large natural number, then to calculate, as in Example 1, the corresponding values of *f′(x),* to plot the points (*x, f′(x)*) in the existing coordinate system and finally to join all these points with a continuous line. The above process is illustrated in the following Example 2.

Example2. In the following figure the graph of a function *f(x)* (red line) is presented.in the interval [-3,3]. Design in the same interval and in the same coordinate system the graph of the derivative *f′(x).*

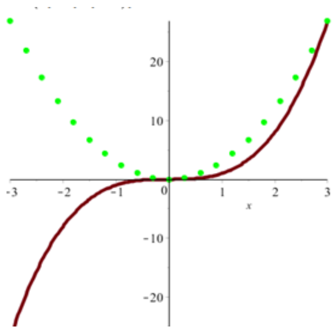

Fig. 3. Graph of f(x) in [-3, 3] and representation of the points of f′(x)

Solution. For symmetry reasons (and only for this) we choose the points with x-coordinates equal to $\alpha+\frac{k(b-a)}{n}$, $\kappa=0, 1,\ldots, n$, where in our case $a=-3$ and $b=3$. If we take, for example $n=20$, the corresponding x-coordinates are -3, $-3+\frac{6}{20}=-\frac{54}{20}$, $-3+\frac{12}{20}=-\frac{48}{20}$,..., $-3+\frac{19.6}{20}=\frac{54}{20}$, 3.

The 21 green points of the graph of *f′(x)* shown in Fig.3 were plotted using the ***Maple*** software.

Further, for $n = 200$ and with the help of Maple one obtains the graph of Fig. 4 (green line).

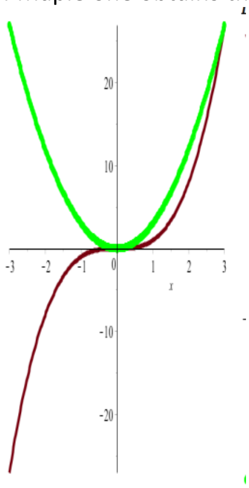

Fig.4. *Graph of f′(x)*

Note. Fig.3 represents the graph of the function $f(x)=x^3$.

Our last Example 3 illustrates another approximate method for designing the graph of *f′(x)* given the graph of *f(x)*, which can be applied with the use of pencil only.

Example 3. The graph of a function $y = f(x)$ is given in the below figure. Observing that its tangent at the point (a, *f*(a)) is horizontal and its tangent at (b, *f*(b)) is vertical with respect to the x-axis, sketch the graph of the derivative function *f′(x)*.

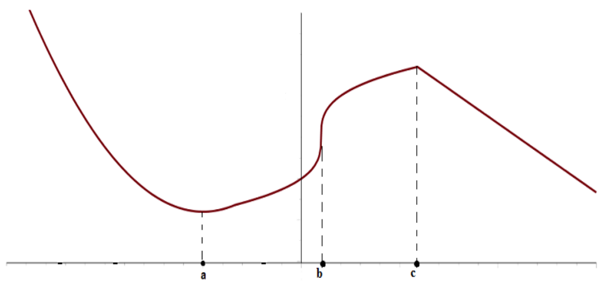

Fig.5. Graph of y = f(x) in Example 3

Solution. Since the tangent of the given graph at (a, *f*(a)) is parallel to the x-axis, its slope is equal to zero, which means that $f'(\alpha) = 0$. Consequently, the graph of *f ′(x)* intersects the x-axis at a.

Also, from Figure 4 one observes that *f(x)* is strictly decreasing in the interval $(-\infty, a)$, which means that *f′(x)<0,* for all x in $(-\infty, a)$. Therefore, the graph of *f ′(x)* in $(-\infty, a)$ lies under the x-axis. Further, the concavity of *f(x)* in ($-\infty$, a), is upwards, which means that $f'''(x)>0$. Consequently, the derivative function $f'(x)$ is strictly increasing in ($-\infty$, a).

In the interval (a, b), *f(x)* is strictly increasing, therefore $f'(x)>0$. Thus the graph of $f'(x)$ lies over the x-axis. Also the concavity of *f(x)* is upwards, which means that $f'(x)$ is strictly increasing

Since the tangent of the graph of *f(x)* at b is vertical its slope is equal to $+\infty$, therefore there is no real value for the derivative of f(x) at b, i.e. b does not belong to the domain of $f'(x)$.

Similarly, in the interval (b, c) we have that $f'(x)>0$ and $f''(x)<0$, i.e. $f'(x)$ is decreasing and its graph lies over the x-axis.

At the point (c, *f*(c)) the left and right tangents to the graph of *f(x)* are different, which means that $f'(x)$ is not defined at c.

Finally, in the interval (c, $+\infty$) *f(x)* is strictly decreasing and its graph turns to a straight line. Therefore the value of the derivative $f'(x)$ is equal to a negative real constant at all points of this interval, which means that its graph is a straight line parallel to the x-axis and lying under it.

All the above lead to the draft design of the graph of $f'(x)$ presented in Fig.6.

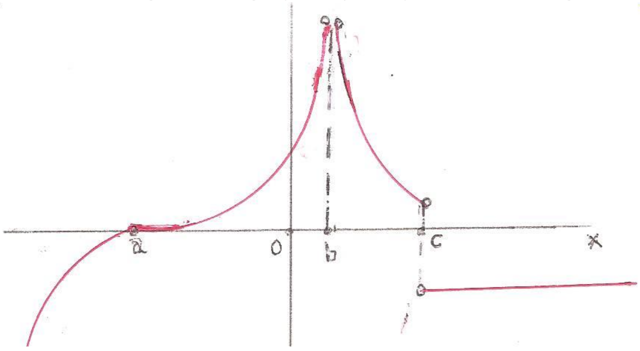

Fig.6. A correct student response for Example 3

From the discussion performed in this paper the following conluson can be drawn:

Conclusion. The graphical representation of the derivatives finds many interesting applications that could and must be reckaimed better for teaching purposes.

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ПРО ГРАФІЧНЕ ПРЕДСТАВЛЕННЯ ПОХІДНИХ

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Анотація. *У статті дано альтернативне визначення поняття похідної, в якому не використовується поняття границі. Це визначення ґрунтується на ідеї Декарта для обчислення нахилу дотичної до кривої в точці і справедливе для всіх алгебраїчних функцій. Каратеодорі розширив це визначення до загального визначення похідної в термінах неперервності. Проте, хоча це визначення успішно використовувалося багатьма німецькими математиками, воно не було широко відомим у міжнародній літературі і не використовувалося у шкільних підручниках. Також у статті описано методи обчислення похідної функції y = f (x) в точці за допомогою підібраної таблиці значень функції y = f (x) та методи побудови графіка похідної функції y = f΄(x) за графіком функції y = f (x), а не за її формулою. Ці методи ґрунтуються на графічному представленні похідної, що можна активно використовувати у навчанні.*

Ключові слова: *вивчення похідної, дотична до графіка функції в точці, алгебраїчні і трансцендентні функції, визначення похідної за Каратеодорі, графік похідної функцій, програма Maple.*