Trajectory of Earnings Growth Influences Cost of Equity Capital, and Optimal Time to Sell

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Abstract

Living individuals and populations of individuals follow an S-shaped growth curve (also known as logistic) from birth to death. A long-term perspective on business firms suggests that they are not exempt from the pattern of a slow growth phase, followed by one of linear to exponential growth, and finally a steady state or growth decline phase. Growth depends on a number of things. Two that are important to the trajectory of growth are the size of the market for the firm's products and the saturation level already reached by its products. New or markedly improved products may, in effect, reset the growth pattern and begin the trajectory once again. That is analogous to a star baseball player continuing to hit 400 or better year after year. This paper models the effects and implications of logistic growth by corporation earnings, and it suggests there is an optimal time for investors to bail out of a particular company's shares. This also has implications for estimating the firm's cost of capital. In contrast to the enduring and popular dividend capitalization model, the analysis presented here is applicable to firms that do not pay dividends. Microsoft provides focus to demonstrate application of the theory developed in the paper on a widely held and followed company.

Introduction

In nature, virtually all organisms and populations follow an S-shaped growth curve (also known as logistic) from birth to death. For example, the growth rate of the volume of wood in a tree increases slowly at first, then accelerates, becomes constant for a while, decelerates as the tree matures and finally can even become negative as damage occurs and decay sets in. This has implications for the determining the optimal time to harvest a plantation of trees. Whole populations of animals and insects may similarly follow an S-shaped trajectory: If one starts with two fruit flies of opposite sex in a sealed jar containing an apple, the population will increase slowly at first, then accelerate, after which it will increase at a decreasing rate, and finally growth will become negative as food runs out or disease ravages the population.

Shareholders value their equity investments for the cash flows the stocks are expected to yield. Shares that currently pay no dividend must be expected to eventually pay dividends or other cash flows or they would have no value. The literature on finance typically[3] offers examples of one, two or – at most – three growth stages, between which the cost of equity capital varies. Invariably, at the final stage, all those textbook examples assume constant growth in perpetuity. However, a constant growth assumption is not reasonable when it implies that a company can continue to indefinitely expand, especially at a rate greater than that of the economies in which it operates. There is ample evidence that earnings growth of corporations often follows the an S-shaped curve, yet textbooks side-step such issues. Journal literature offers little in the way of contrary advice or alternatives to the dividend capitalization model (DCM), often termed the Gordon (1962) model. One exception by inference is Haugen (1999) who challenges the concept of growth as it is normally considered to exist in companies.

Attempts over the years to incorporate growth patterns other than constant-rate growth have not adequately addressed the question of what model best reflects the reality of growth. A recent article by Haslem (2002) is representative of what has been done. Among the works relevant to our study he cites earlier works by Bierman (2001), Williams (1938), and Brigham and

Pappas (1966). Haslem correctly points out that dividend models are not very useful in valuing stocks that pay little or no dividend. However, he follows precedent by basing his analysis on three discrete stages (pioneering, investment maturity, and stabilization) and does not try to formally incorporate a logistic function. He correctly mentions in his conclusion that the particular model used to describe the dividend time path has a marked influence of the share price value.

The trajectory of corporate earnings growth is important because it affects the firm's cost of equity capital and thus its overall cost of capital. This has implications for capital investment decisions based on discounted cash flow estimates as well as for investors. The trajectory of corporate earnings growth is important to investors because equity share value should mirror changes in earnings, and consequently in dividends currently paid and in prospect to be paid in the future.

In the dividend capitalization model popularized by Gordon (1962), stock price is the time-zero present value of all future cash flows; that is, cash dividends. This means that stock price is entirely dependent on dividends, extant and in prospect, as discounted at the market's (subjective) discount rate. We extend this by assuming that stock price P_t is time-dependent – that earnings and dividends depend on time – and they are the main determinants of stock value. With a logistic trajectory, the growth rate eventually slows. Thus, eventually the curve of present value of earnings and dividends plotted against time will begin to bend downward, and the point of its maximum is where one should bail out of the stock.

There is ample evidence of S-curves, not only in nature, but in business as well. Van Duijin (1983, p. 22) avers that the logistic is the backbone of the product-life-cycle in literature on marketing. The term product-life-cycle was originated by Joel Dean (1950). Dean identified three stages of a product's life: introduction, growth, and maturity. He claimed that the length of the cycle is determined by the rate of technological change. Van Duijin mentions that four, and even five phases, are now distinguished, with the last stage termed decline.

Logistic growth applies to whole companies and whole industries, not just products and services they manufacture and sell. Miller (1990) provides a compelling and lucid rationale for why companies may follow an S-curve from birth to their eventual demise. He argues that companies eventually decline and fail when they fail to adapt to changing conditions by continuing to operate in the same way that gave them success in the past. Rigid adherence to the tried and true, refusal to adapt, when markets, technologies, and competition change, is the path to certain decline and eventual ruin if not corrected by management. Foster and Kaplan (2001, p. 55) make a case for the S-curve.

> *"While this pattern may vary somewhat for sales or earnings—for example, near the top of the S-curve it is not uncommon for sales to go into a cyclical period, or turn down rather than stay constant—it is a reasonable approximation in all industries we have studied."*

They continue by pointing out that early long-term forecasts of sales or earnings will be too low, and later on too high, requiring continual adjustment. At the beginning stage of upward revisions the stock price will rise, as will returns to investors. But at the later stage, after the growth has peaked, downward revisions will have the opposite effects. Foster and Kaplan (2001, p. 57) point out that

> *"This is exactly what happens in fast-moving industries like computer hardware. During the period from 1962 to the mid-1970s, both the total return to shareholders and the price-to-earnings ratio of the computer industry were above the corresponding levels in the US economy as a whole. This proved to be a nightmare for value investors who shunned this industry waiting for a better time to invest. When the 'better time' came, the companies were underperforming the market . . . until a new group of leaders took over in the market and drove performance up again by riding the new S-curve of personal computers and eventually the Internet."*

Readers can find numerous examples of S-curve growth in their book.

Stewart (1989) makes a case for forecasting a company's future by taking measurements along its past trajectory, and then applying those measurements to extrapolating the future evolution along a logistic curve. His focus is more on industries and macroeconomic measures than on individual firms, yet may be applicable to them as well.

In this paper we present results from integrating S-shaped growth into the dividend capitalization model. First, we examine the theoretical implications of varying growth on share price value. Then we examine a company whose data indicate fits the S-shaped growth model, Microsoft[1] (ticker symbol MSFT). In doing so we fit a sigmoid curve to the company's earnings and measure the extent to which investors are likely to err if they value the firm's shares according to the Gordon form of the model instead of one that reflects varying growth. The example is used solely to illustrate application of the theory, which does not depend on any particular instance in a company.

S-Shaped Functions

Several specific mathematical forms of S-shaped curves form a class collectively termed sigmoids. Two of the more popular forms are the logistic curve and the Gompertz curve. The logistic, often called the Pearl-Reed curve, or simply the Pearl curve, is somewhat simpler. It assumes a symmetrical function, while the Gompertz does not require symmetry. We adopt the logistic here for purposes of exposition. Martino (1983, p. 58-59) provides insight into a key difference between the two characterizations:

> *"The Pearl and Gompertz curves have completely different underlying dynamics. This can be seen by taking their derivatives: The slope of the Pearl curve is proportional to y(L − y) and the slope of the Gompertz curve is (for y > L/2) proportional to L − y only. That is, the slope of the Gompertz curve is a function only of distance to go to the upper limit, whereas the slope of the Pearl curve is a function of both distance to go and distance already come."*

In analogy to the computer software market, for example, a Gompertz curve would not be affected by the installed base of operating systems and office suite program collections. The Pearl curve would be. Given that much computer software is sold to replace existing operating systems and application programs, we believe the Pearl curve better represents the reality of the computer software industry than the Gompertz does. And it may also represent many other industries where replacement is involved better.

The logistic curve itself has several different mathematical representations. Basically, any growth rate $\dfrac{dF(t)}{dt} / F(t)$ that is a decreasing function of the original $F(t)$ is a logistic function. For example, if $F(t)$ is a population at time t, and $a - bF(t)$ is a decreasing function, then a logistic function can be $\dfrac{dF(t)}{dt} \Big/ F(t) = a - bF(t)$ and the solution for this particular logistic function is $F(t) = \dfrac{a}{b + k \cdot e^{-at}}$, where $k = e^{-ak_1}$ and k_1 is a constant of integration.

Another form of logistic growth is

$$\frac{\dfrac{dF(t)}{dt}}{F(t)} = \frac{ab}{e^{bt}+a} \quad \text{with } a,\ b>0\,, \tag{1}$$

which has the solution

$$F(t) = \frac{c}{1+ae^{-bt}} \quad , \tag{2}$$

where $c > 0$ is constant and is the limiting value, or upper asymptote. This particular form is from Raymond Pearl, a US demographer who popularized its use for population forecasting.

Dividend Growth

Gordon (1962) uses a discrete constant dividend growth rate g such that $D_t = D_0(1+g)^t$, where D_t is the dividend at time t. We substitute a continuous logistic growth function for the exponential curve determined by the constant growth rate, g. The difference between discrete and continuous compound growth means such substitution is not entirely analogous, but the comparative statics of the discrete and compound growth curves are comparable and the continuous case is a close approximation to its discrete counterpart.

The following assumption should make the theory sufficiently general to include the case of a growing firm that does not pay dividends. We assume that the investor predicts that the firm's earnings will grow according to a logistic function and also prices the stock as if the dividends were paid as a fixed percentage (e.g. 70%) of earnings.

The actual or anticipated dividend, $D(t)$ at time t, is a function of the dividend payout ratio, δ, and the firm's logistic earnings curve, $E(t)$: $D(t) = \delta E(t)$. We assume that $0 \le \delta \le 1$ on average; that is, the firm will not normally pay a dividend greater than its earnings, though it may choose to pay a constant dividend that exceeds earnings in any given quarter or year.

Let us assume that $D(t)$, the dividend at time t, takes on the logistic growth form as above $F(t)$ in equation 1, namely

$$\frac{\dot{D}(t)}{D(t)} = \frac{ab}{e^{bt}+a} \quad \text{for} \quad a,b>0\,. \tag{3}$$

The solution $D(t)$ will be $D(t) = \delta E(t) = \dfrac{c}{1+ae^{-bt}}$ (with $a,b,c>0$). This suggests that there may be an optimal time to sell this stock investment.

Optimal Timing

Under constant growth, the investor is indifferent between cashing out the dividend stream either by selling out or by just keeping the dividend stream indefinitely. Under constant perpetual growth, going forward from any point t the present value of future dividends is constant. But, if the dividend takes a logistic path, then the investor will want to find the optimal time to sell the stock, retrieve invested capital, and possibly invest in another stock that is still in the growth portion of its own logistic path. If the growth rate is not constant, but eventually declines, there will be a time at which the present values of future earnings and dividends decline – that is the optimal bail out point[1]. Our model has a subjective discount rate, and thus the present values can be different for different investors, and everyone may have a different optimal bail out time, just as all market participants may hold different expectations in order that there are both buyers and sell-

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[1] Selling depends on our discounted curve, not on the deceleration phase. If the investor's subjective discount rate is high, optimal selling could even be in the acceleration phase. The fact that a stock's return follows a logistic path means we should not buy and hold, but should consider that there is an optimal time to sell, and that time is dependent on the subjective discount rate.

ers at the same time. Our use of the continuous growth curve implies we are assuming the investor will calculate dividends on a continuous and not discrete basis. We define optimal time $t_1* \equiv \max\limits_{t} e^{-rt}D(t)$ and get

$$t_1^* = \frac{\ln\left[\dfrac{ab - ar}{r}\right]}{b}, \qquad (4)$$

with corresponding

$$D\left(t_1^*\right) = c - \frac{rc}{b}. \qquad (5)$$

In other words, t_1^* is the solution for the above optimization problem that meets the usual first and second order conditions. The term e^{-rt} has the usual meaning of the continuous discounting factor based on the (in this case subjective) discount rate, r. The term b determines the slope $\dot{D}(t)$. We can look at b, which affects the slope, as a factor that marks the rapidity with which the firm matures, that is, penetrates all its markets and reaches peak sales and earnings.

The greater the b term (the steeper the growth curve in dividend/anticipated dividend), *ceteris paribus*, the quicker that the optimal abandonment horizon, t_1^*, will be reached. There are three possible cases, namely $b > r$, $b = r$ and $b < r$. The Appendix shows that $b > r$ is the only case where both the first and second order conditions are satisfied. Please refer to Figure 1A[1].

In Figure 1A, if the dividend traces out an S-curve, then at rate r, the discounted curve will bend downwards implying that:

1. At the inflection point of the discounted dividend curve, it is time to sell shares. This is in contrast to Gordon's hold-forever thesis or that value is constantly growing.
2. If one does not sell at the inflection point that would imply that the rate r for periods after the inflection point must decrease such that the discounted values do not bend down the dividend curve.
3. Alternatively, one might expect some influences (e.g., successful new products of the company or drastic cost cutting) that could change the logistic earnings and dividend functions[2].

Otherwise, holding beyond the optimal abandonment horizon point t_1^* implies that the investor is irrationally willing to stay with his or her investment in a company beyond the point where doing so provides a positive expected value change from the investment. An investor would not do so unless he or she had unrealistic expectations about the company's future performance.

To illustrate with a numerical example, Figure 1A is graphed with the following parameters:

Asymptote $c = 1000$ cents; time $= 100$ periods;
Slope determinant $b = 0.10 > r = .00745$ per period;
Since the curve is symmetrical over [0,100], the inflection point lies at
$t = 50 = \ln[a] / b \Rightarrow a = \exp(50 \times b) = \exp(5) = 148.41$

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[1] Some may object that the logistic parameters can only be determined ex post. However, it is possible to estimate them early in the process. For an intuitive sense of how this might be approached see Hugh B. Stewart, Recollecting the Future, (1989).

[2] It is sometimes the case that one logistic growth trajectory is followed by a second and possibly even more. This requires that old patterns of growth be replaced by new innovations as the old reach saturation or decline. For more on this see Perrin Meyer, "Bi-Logistic Growth", which first appeared in the journal Technological Forecasting and Social Change, published by Elsevier Science Inc., New York.
URL: http://phe.rockefeller.edu/Bi-Logistic/ Citation: Technological Forecasting and Social Change 47:89-102 (1994).

Fig. 1A. Logistic Growth ($r < b$)
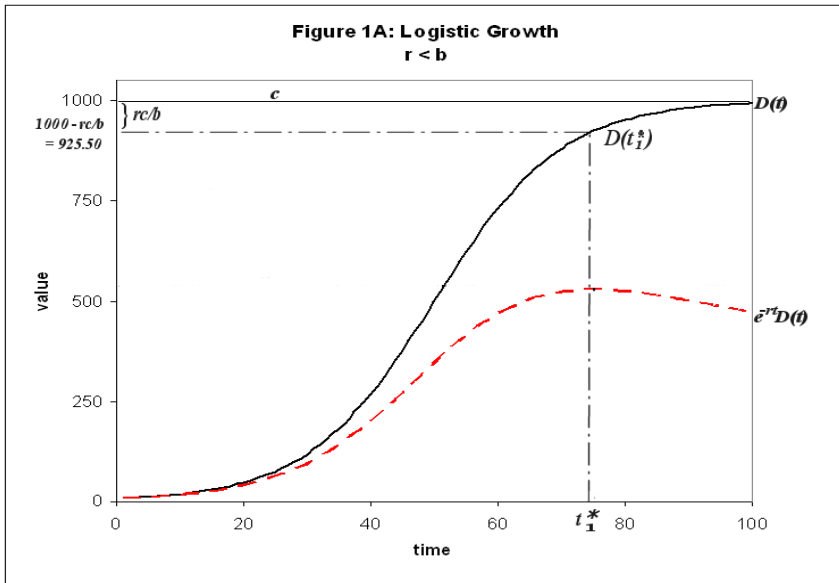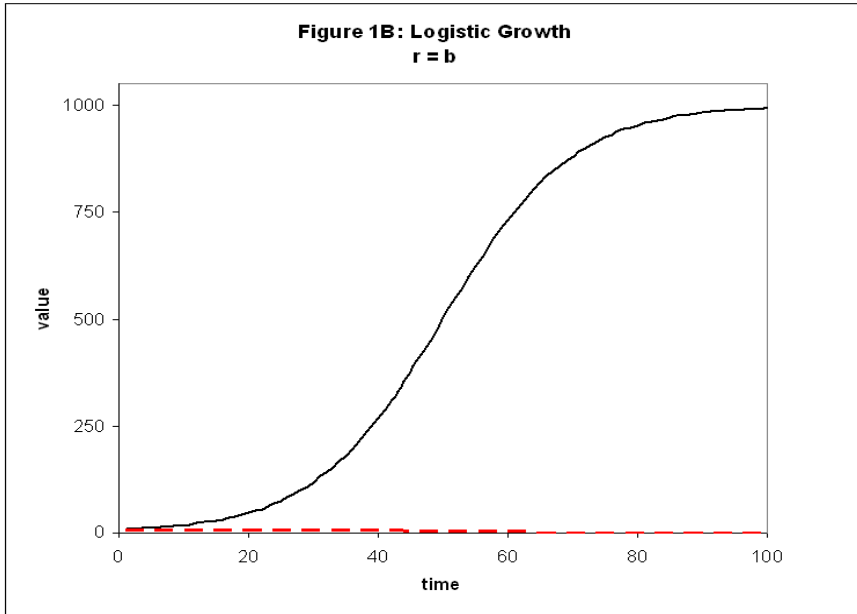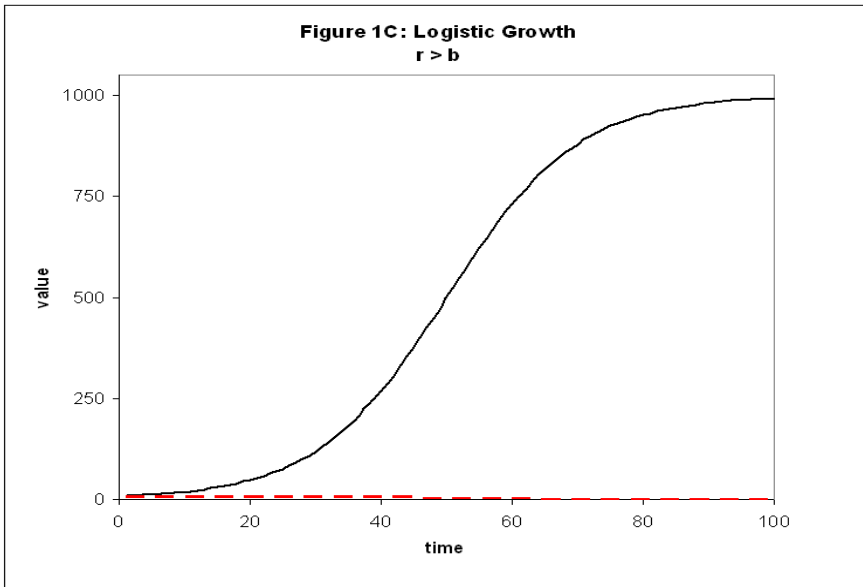

Fig. 1B. Logistic Growth ($r = b$)

Fig. 1A. Logistic Growth ($r > b$)

Therefore the optimal timing when the discounted dividend $e^{-rt}D(t)$ is maximized is

$$t_1^* = \ln[\,a(b-r)/r\,]/b = \ln[1843.67]/0.10 = 75.1951^{\text{th}} \text{ period}^1 \text{ and } D(t_1^*) = c - rc/b = 1000 - 74.5 = 925.50.$$

For the other two cases of b = r and b < r, the second order condition is not met and we do not have a solution for the optimal timing problem. The economic interpretation is quite clear – when the subjective discount rate r is greater than or equal to the growth determinant, the investor should not be willing to buy the stock. Please see Figures 1 B and 1 C, which are drawn with parameters r, b set at r = b and r > b, respectively. For the case of r > b the difference between them is only 0.0001, 0.10001 versus 0.10000, yet even for this minor difference the curve collapses.

Application to MSFT

A company widely followed by investors and the media is Microsoft, ticker symbol MSFT. Microsoft, founded in the 1980s, grew to maturity in less than twenty years. Its diluted earnings per share became positive in the first quarter of 1988 (Q88-1 in the notation we use here). A period of rapid increase and then a recent slowing of growth followed the slow initial growth in earnings per share. This suggests that a logistic might be well suited to representation of Microsoft's earnings growth trajectory. We use MSFT diluted per-share earnings from the first quarter of 1988 to the third quarter of 2001 inclusive, thus ignoring all the quarters of zero reported earnings before 1988. We also make the assumption that the anticipated dividend will be 100% of the diluted earnings (i.e., the payout ratio, $\delta = 1$) to simply the analysis.

We fit the simple Pearl form of logistic curve discussed above to MSFT. Figure 2 shows the results of fitting the curve to MSFT earnings. We do not include results of experimentation with the Gompertz curve. They tend to confirm our aforementioned assumption that it is not as well suited as the logistic curve to Microsoft.

Note again that the fitting process does not use the most recent data points. It would appear they are either outliers or else Microsoft has experienced a structural change to its earnings.

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[1] We note that the higher the subjective discount rate r (still < b), the shorter the time to bail-out is. For example, if we increase r to 9% per period, $t_1^* = 28.03$th period.

The sum of the squared error terms for the Pearl logistic curve is 0.22304. The calculated asymptote is \$3.13 for the time span Q88-1 through Q01-3 inclusive[1]. During this time the maximum diluted per-share earning for MSFT was \$1.75 in Q01-3. Following that quarter it has fallen off markedly.

The parameters obtained in fitting the above logistic to MSFT quarterly data[2] (shown in Figure 2) are: asymptote c = \$3.13; b = 0.1155; inflection point ln $[a]$ / b = 63.09504. Hence a = exp (7.2875) = 1461.91.

Further, if we assume that the investor's subjective discount rate r = 0.09 per annum ($r <$ b): Optimal bail-out time $t_1^* = \ln [a(b-r) / r] / b = \ln (414.21) / 0.1155 = 52.18^{th}$ quarter, or between Q2001-1 and Q2001-2.

Fitting the upper asymptote to a Pearl logistic curve presents a conceptual problem. Martino (1983, p. 61) advises against approaches that optimize the curve fit by finding the asymptote as a part of the fitting process.

> *"Some forecasters use curve-fitting methods that extract from the historical data not only the two coefficients of the Pearl or Gompertz curve, but the upper limit L as well. This is bad practice and it should not be done. During the early history of a technical approach the upper limit has very little effect on its growth in performance. Thus data points from this period contain little information on the upper limit. Values for L obtained by such means are certain to contain a large error component.*
>
> *Instead, the upper limit should be estimated on the basis of, for example, the physical and chemical limits imposed by nature on the technical approach to be forecast. These natural limits may exist in the form of a breakdown voltage, a maximum efficiency, limiting mechanical strength, a maximum optical resolution, a minimum detectable concentration of a chemical, and so on."*

We proceed contrary to this advice because, unlike the applications Martino addressed, we are aware no logical, a priori method that will provide an upper limit to Microsoft's diluted earnings per share that is not wholly arbitrary. Therefore, we choose to determine the upper asymptote from the data we have, rather than from other methods aimed at determining the maximum per-share earnings MSFT approaches in the limit. The calculated asymptote is \$3.13 for the time span Q88-1 through Q01-3 inclusive[3]. During this time the maximum diluted per-share earning for MSFT was \$1.75 in Q01-3. Since that quarter it has fallen off markedly.

The alternative logistic, based on fitting with c held to \$1.76 is: b = 0.188835, a = exp(54.08637 x 0.188835) = exp(10.21340) = 27,266.11, and r = 0.09 per annum yields

t_1^* = ln (29,942.73) / 0.188835 = 54.58, between Q2001-2 and Q2001-3.

The graph does not look appreciably different from that in Figure 2 so we do not include it here.

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[1] An asymptote of \$1.76, slightly above the maximum realized per-share earnings offers a better fit to the shape of the earnings curve in recent quarters, but a poorer fit to the early data. We fitted the logistic by minimizing the squared deviations from the realized diluted earnings per share.

[2] We note that the subjective discount rate r in the logistic function is normally a per annum figure, as it is in the usual continuous compounding formula, e.g. ert. What use here the MSFT quarterly diluted earnings as data points to trace out an earnings-anticipated-dividend curve. We then fit a logistic function to approximate this earnings-dividend curve. And the r in the fitted logistic function is on a per annum basis.

[3] An asymptote of \$1.76, slightly above the maximum realized per-share earnings offers a better fit to the shape of the earnings curve in recent quarters, but a poorer fit to the early data. We fitted the logistic by minimizing the squared deviations from the realized diluted earnings per share.
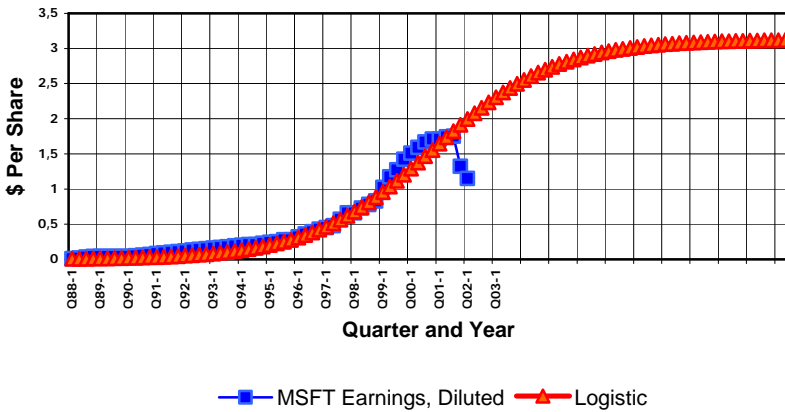
Fig. 2. MSFT Earnings Per Share

Table 1 contains optimum bailout numbers for various values of *r* and asymptotes of $3.13 and $1.76.

Table 1

Optimal Bailout for Various *r* and asymptotes of $3.13 and $1.76 per quarter earnings per share

*t** ($3.13)	*r*	*t** (1.76)	*t** ($3.13)	*r*	*t** (1.76)	*t** ($3.13)	*r*	*t** (1.76)
1	2	3	4	5	6	7	8	9
104.13886	0.10000%	81.81198	66.35644	4.70000%	59.93548	50.80881	9.30000%	54.24538
98.06167	0.20000%	78.11306	66.04683	4.80000%	59.78652	50.32260	9.40000%	54.13319
94.47454	0.30000%	75.93745	65.73909	4.90000%	59.63959	49.81863	9.50000%	54.02102
91.90649	0.40000%	74.38542	65.43299	5.00000%	59.49460	49.29499	9.60000%	53.90883
89.89652	0.50000%	73.17500	65.12834	5.10000%	59.35145	48.17958	9.80000%	53.68430
88.23927	0.60000%	72.18061	64.82493	5.20000%	59.21006	47.58225	9.90000%	53.57191
86.82521	0.70000%	71.33524	64.52258	5.30000%	59.07034	46.95395	10.00000%	53.45941
85.58893	0.80000%	70.59890	64.22110	5.40000%	58.93222	46.29041	10.10000%	53.34677
84.48825	0.90000%	69.94580	63.92030	5.50000%	58.79563	45.58643	10.20000%	53.23396
83.49436	1.00000%	69.35832	63.61999	5.60000%	58.66050	44.83566	10.30000%	53.12095
82.58671	1.10000%	68.82390	63.32000	5.70000%	58.52675	44.03012	10.40000%	53.00773
81.75012	1.20000%	68.33326	63.02015	5.80000%	58.39433	43.15966	10.50000%	52.89426
80.97305	1.30000%	67.87935	62.72025	5.90000%	58.26317	42.21111	10.60000%	52.78052
80.24655	1.40000%	67.45670	62.42012	6.00000%	58.13322	41.16687	10.70000%	52.66647
79.56348	1.50000%	67.06096	62.11959	6.10000%	58.00442	40.00273	10.80000%	52.55210
78.91813	1.60000%	66.68864	61.81847	6.20000%	57.87672	38.68404	10.90000%	52.43737
78.30579	1.70000%	66.33686	61.51658	6.30000%	57.75007	37.15873	11.00000%	52.32226
77.72257	1.80000%	66.00326	61.21373	6.40000%	57.62442	35.34312	11.10000%	52.20673
77.16520	1.90000%	65.68585	60.90972	6.50000%	57.49973	33.08983	11.20000%	52.09076
76.63091	2.00000%	65.38295	60.60437	6.60000%	57.37594	30.10011	11.30000%	51.97431
76.11735	2.10000%	65.09312	60.29748	6.70000%	57.25302	25.60209	11.40000%	51.85736
75.62248	2.20000%	64.81512	59.98883	6.80000%	57.13092			

Table 1 (continuous)

1	2	3	4	5	6	7	8	9
75.14452	2.30000%	64.54788	59.67822	6.90000%	57.00960			
74.68193	2.40000%	64.29047	59.36542	7.00000%	56.88902			
74.23335	2.50000%	64.04207	59.05020	7.10000%	56.76916			
73.79758	2.60000%	63.80195	58.73233	7.20000%	56.64996			
73.37354	2.70000%	63.56946	58.41155	7.30000%	56.53139			
72.96028	2.80000%	63.34405	58.08760	7.40000%	56.41343			
72.55695	2.90000%	63.12519	57.76021	7.50000%	56.29603			
72.16275	3.00000%	62.91243	57.42907	7.60000%	56.17916			
71.77700	3.10000%	62.70534	57.09389	7.70000%	56.06279			
71.39905	3.20000%	62.50355	56.75432	7.80000%	55.94689			
71.02831	3.30000%	62.30672	56.41001	7.90000%	55.83144			
70.66426	3.40000%	62.11454	56.06059	8.00000%	55.71639			
70.30640	3.50000%	61.92672	55.70566	8.10000%	55.60172			
69.95427	3.60000%	61.74300	55.34476	8.20000%	55.48741			
69.60745	3.70000%	61.56314	54.97743	8.30000%	55.37341			
69.26556	3.80000%	61.38692	54.60316	8.40000%	55.25972			
68.92822	3.90000%	61.21414	54.22139	8.50000%	55.14629			
68.59510	4.00000%	61.04460	53.83151	8.60000%	55.03310			
68.26587	4.10000%	60.87814	53.43284	8.70000%	54.92013			
67.94024	4.20000%	60.71459	53.02464	8.80000%	54.80735			
67.61791	4.30000%	60.55379	52.60612	8.90000%	54.69473			
67.29861	4.40000%	60.39561	52.17634	9.00000%	54.58225			
66.988210	4.50000%	60.23991	51.73431	9.10000%	54.46988			
66.6612	4.60000%	60.08657	51.27890	9.20000%	54.35760			

In population forecasting models, researchers usually linearize the logistic function by the natural logarithm function, ln, and use regression analysis to determine the parameters *a* and *b*. These parameters are then plugged into the logistic function and extrapolated for future population estimates. We mention this as a possibility for future research since it is outside the scope of this paper.

Another possible direction for future research will be the application of logistic growth analysis in picking stocks. Our analysis suggests the following strategy. If the investor expects logistic growth for a set of small companies with promising new products or services, he or she will invest in some of these companies. Individual companies will fail. But the expectation is that as a group they will grow. Thus the investor should enjoy the rapid growth portion of the logistic curve for those that are successful. The investor loses nothing by waiting for the inflection point on *D(t)* to appear for any individual company since the actual or anticipated dividend is rising on the rapid growth portion of the curve. When realized growth exhibits a pattern resembling the portion beyond the inflection point, the investor should apply the above optimal timing analysis to estimate the optimal bailout point. In case the firm's earnings-dividend growth pattern is already on the rising portion of the logistic curve, the investor can still estimate the *a, b, c* parameters and calculate optimal bail-out time. From the optimal bail-out time, the investor can infer the remaining possible anticipated dividend gain and hence the return based on the current stock price. The decision of whether to invest then depends on an estimation of the firm's ability to grow according to the logistic function and on a subjective risk-return preference curve.

Logistic growth has implications for cost of capital that make estimation of it by corporation management more challenging. According to the above analysis and the illustrative application of

logistic growth, the investor's subjective rate is compared to *b*, which determines the slope of the growth curve. Equation 5 can be used to solve for *r*, the investor's minimum required rate of growth.

$$r = b\left[1 - \frac{D\left(t_1^*\right)}{c}\right]. \qquad (6)$$

In terms of the dividend expected at the optimal abandonment time, the asymptote of the logistic growth, *c*, and the slope-determining parameter, *b*. Given the caveats concerning estimation of the asymptote mentioned above, this poses difficulties for management that seem less easily resolved than those associated with application of the CAPM or dividend capitalization model for estimating cost of capital. However, the dividend capitalization model is inapplicable to stocks that pay no cash dividend. And the CAPM approach depends on stability in the relationship between the returns on a stock and those on the market that cannot exist if the company's growth changes with respect to the market over time. Therefore, the cost of capital given by equation 6 may still be superior for companies that pay no dividend and follow a logistic growth trajectory.

Conclusion

In this paper our approach is to adapt to the firm a logistic growth model widely used in the physical sciences. Previous research on the theory of the firm has considered only one, two, or at most three discrete phases of a firm's growth to approximate the firm's cost of capital (or value) using the now traditional dividend capitalization (Gordon) model. We show that if the earnings-implied anticipated dividend of a firm were to follow a logistic curve, there is an optimal time for the investor to sell. In contrast, the Gordon model, assumes earnings and dividends to reach a point of constant growth that lasts indefinitely into the future and thus there is no optimal point to sell.

We suggest that future research might examine the influence that widespread adoption of the model presented in this paper could have on the pricing of common stock. In the meanwhile, it offers individual investors a means for gauging whether to continue to hold or to sell shares based on theory that recognizes non-constant growth.

We also suggest that the problem of estimating a firm's cost of capital by using the results from the logistic curve model is not demonstrably more difficult than those in applying the CAPM. It is a matter of a trade-off of one difficulty for another, an unstable beta for a problematic asymptote value, for instance. And, cost of capital can be estimated by using the results of logistic curve modeling, while the dividend capitalization model cannot be sensibly used at all for stocks that do not pay dividends.

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Appendix

Constant Growth (Gordon) Model

$\hat{D}_t$ = expected dividend at time t

$\hat{k}_e$ = subjective expected rate of return

P_0 = price of stock

$\hat{g}$ = expected constant growth rate of dividend

D_0 = dividend at time 0

We conjecture that $P_0 = \sum_{i=1}^{n} \dfrac{\hat{D}_t}{\left(1 + \hat{k}_e\right)^t}$ $n \to \infty$

and that $\hat{k}_e > \hat{g}$

As $n \to \infty$ $P_0 = \dfrac{D_1}{\hat{k}_e - \hat{g}}$ or $\hat{k}_e = \dfrac{D_1}{P_0} + \hat{g}$

But, a constant growth rate forever is unrealistic and has never been observed. Thus, we replace the growth rate with the more realistic logistic curve.

One form of logistic growth $\dfrac{\dot{D}}{D} = \dfrac{ab}{e^{bt} + a}$ $a, b > 0$ has the solution

$D(t) = \dfrac{c}{1 + ae^{-bt}}$ $a, b, c > 0$ and has the properties:

1. A. The curve is growing, but at a continuously decreasing rate.

As $t \uparrow \Rightarrow$ denominator $(e^{bt} + a) \uparrow \Rightarrow \dfrac{ab}{(e^{bt} + a)} = \dfrac{\dot{D}}{D} \downarrow$

B. Parameter c is an asymptote, or saturation level

As $t \to \infty$ $e^{bt} \to 0$ $\Rightarrow D(t) = \dfrac{c}{1 + a(\to 0)} \to c$

2. $a \uparrow$ shifts the curve to the right. $(a \uparrow \Rightarrow$ inflection point $\dfrac{\ln(a)}{b} \to$ right$)$

3. Parameter b determines the slope of the logistic curve $= \dot{D}(t) = \dfrac{abce^{-bt}}{(1 + ae^{-bt})^2}$.

4. Inflection point lies at $t = \dfrac{\ln(a)}{b}$ and $D(t) = \dfrac{c}{2}$

(Inflection is at ½ the saturation level, and the curve is symmetrical with respect to the inflection point).

5. D-intercept is $D_0 = \dfrac{c}{1 + a(1)} = \dfrac{c}{1 + a}$

Optimal timing:

Find t_1^* that yields Max(Present Value of $D(t)$), i.e.,

$$\underset{t}{\text{Max}}\left[PV(t) = e^{-rt}\left(\dfrac{c}{1 + ae^{-bt}} \right) \right]$$

$$\ln[PV(t)] = \ln\left[e^{-rt}\left(\frac{c}{1+ae^{-bt}}\right)\right]$$

$$= \ln(e^{-rt}) + \ln\left(\frac{c}{1+ae^{-bt}}\right)$$

$$= -rt + \ln(c) - \ln(1+ae^{-bt})$$

$$\Rightarrow \frac{d\ln[PV(t)]}{dt} = \frac{1}{PV(t)}\frac{dPV(t)}{dt} = -r + 0 - \frac{ae^{-bt}(-b)}{1+ae^{-bt}}$$

$$\Rightarrow \frac{dPV(t)}{dt} = PV(t)\left[-r + 0 - \frac{ae^{-bt}(-b)}{1+ae^{-bt}}\right] = e^{-rt}\frac{c}{1+ae^{-bt}}\left[-r + 0 - \frac{ae^{-bt}(-b)}{1+ae^{-bt}}\right]$$

Now, set $\dfrac{dPV(t)}{dt} = 0$ for the first order condition:

$$0 = -\left(\frac{e^{-rt}cr}{1+ae^{-bt}}\right) + \left(\frac{e^{-rt}cabe^{-bt}}{(1+ae^{-bt})^2}\right)$$

$$\Rightarrow \boxed{r = \frac{abe^{-bt}}{1+ae^{-bt}}}$$

$$\Rightarrow r + rae^{-bt} = abe^{-bt}$$

$$\Rightarrow r = abe^{-bt} - rabe^{-bt} = e^{-bt}[ab - ar]$$

$$\Rightarrow e^{bt} = \frac{a(b-r)}{r} \Rightarrow bt = \ln\left[\frac{a(b-r)}{r}\right]$$

$$\Rightarrow t_1^* = \frac{\ln\left[\dfrac{a(b-r)}{r}\right]}{b} = \boxed{\frac{\ln(a)}{b} + \frac{\ln(b-r)}{b} - \frac{\ln(r)}{b}}$$

Now the second order condition must also be satisfied:
Case 1: $b > r$ satisfies 2nd order condition, but
Case 2: $b \le r$ does not necessarily satisfy 2nd order condition.
Second order condition:
At critical point $t_1^* \Rightarrow \dfrac{dPV(t)}{dt} = 0$

$$\Rightarrow \frac{d^2PV(t_1^*)}{dt^2} = \frac{ab}{c}\left[PV^2(t)e^{(r-b)t}(r-b) + 0\right]$$

$$\Rightarrow \frac{d^2PV(t_1^*)}{dt^2} < 0 \text{ if } (r-b) < 0 \Leftrightarrow r < b$$

To find $D(t_1^*)$ we know

$$bt_1^* = b\left\{\frac{\dfrac{\ln[a(b-r)]}{r}}{b}\right\} = \ln\left[\frac{a(b-r)}{r}\right]$$

$$\Rightarrow e^{bt_1^*} = \frac{a(b-r)}{r} \Rightarrow e^{-bt_1^*} = \frac{r}{a(b-r)}$$

$$\Rightarrow D(t_1^*) = \frac{c}{1+ae^{-bt_1^*}} = \frac{c}{1+a\left\{\dfrac{r}{a(b-r)}\right\}}$$

$$= \frac{c}{1+\dfrac{r}{(b-r)}} = \frac{c}{\dfrac{b-r+r}{b-r}} = \frac{bc-rc}{b}$$

$$= c - \frac{rc}{b}$$