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NUMERICAL ANALYSIS OF DYNAMICAL SYSTEMS AND THEIR STRUCTURAL STABILITY

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The method for investigating the dynamics of concrete systems of small dimension and obtaining strict results is demonstrated on the example of M. Henon's system. The program realized as the C# – application and with usage of technology Open Maple is used here. The program allows to discover strange attractors for dynamical systems and to prove the hyperbolic dynamics on them, using outcomes of evaluations on the computer. However we get the strict a posteriori results here based on theorems of the article while the numerical evaluations are used only for checking the validity of assumptions of these statements.

A structural stability of the model leads to a possibility of mathematically justified numerical analysis. It is the based concept of two traditional university courses: "Mathematical modeling and system analysis" and "Methods of calculations". This article is an introduction to a solution of this problem proposed by the author. It became clear that for this purpose it suffices to consider the dynamics with an explicit account of unavoidable random fluctuations. More precisely, for a given classical system we construct its perturbation by a Markov process called a dynamic quantum model (DQM). The structurally stable realizations of DQM are dense everywhere, that allows one to investigate DQM by numerical evaluations. On the other hand, as the fluctuations tend to zero, the results obtained for DQM become statements about initial classical dynamics.

Keywords. *Dynamical, systems, quantum, structural, theory, algorithm, attractor.*

Introduction. Stability of a mathematical model with respect to small variation of the parameters is a necessary condition of its correctness. If an arbitrarily small perturbation may lead to a qualitatively different picture of dynamics, then such a model is not applicable to the real process investigated experimentally. Strictly speaking, errors are included in the model by definition. Neither numerical analysis nor computing experiment is applicable to unstable models as there are inevitably sampling and rounding off errors.

The qualitative invariance of a mathematical model under small perturbations is usually called structural stability. This formally means equivalence, in some exact sense, between the model and its small enough perturbation. For the smooth dynamical systems (set by differential or difference equations) this equivalence is usually a homeomorphism between the phase portraits of these systems. Such theory of structural stability going back to H. Poincare, has been developed by A.A. Andronov and L.S. Pontrjagin in the case of small dimension of the phase space (1 or 2) [1]. However, the optimism generated by the successes of this theory disappeared after S. Smale's works [2]. It was shown in [3] that when the phase space has larger dimension, then there exist smooth dynamic systems which neighborhoods do not contain any structurally stable system. For the theory of smooth dynamical systems (its old name is the qualitative theory of the differential equations) this result has the same value as Liouville's theorem on insolvability of the differential equations in quadratures as for the theory of their integration. Namely, it shows that the problem of full topological classification of smooth dynamical systems is hopeless. This meant that there was no strict mathematical basis for modeling and the numerical analysis of systems in general position. This is a contradiction in a science since the physicists believe that the dynamics should be arranged simply and universally.

This article is an introduction to a solution of this problem proposed by the author. As a matter of fact, it became clear that for this purpose it suffices to consider the dynamics with an explicit account of unavoidable random fluctuations. But really only such dynamics is given to us in experiment and evaluations. More precisely, for a given classical system we construct its perturbation by a Markov process called a dynamic quantum model (DQM) [4]. For Hamiltonian systems, there is a simple connection between these Markov processes and the quasisolutions of the corresponding Schrodinger equation, while construction of DQM is a method of solution of spectral problems of quantum mechanics [5].

However, as a matter of fact, DQM is not connected with Hamiltonian systems in any way. It is defined for an arbitrary ordinary differential or a difference equation on any smooth Riemannian manifold. Non-Hamiltonian quantum dynamics obtained this way appears to be easier than the classical one, and allows returning in essence to a simple picture of H. Poincare's dynamics. The structurally stable realizations of DQM are dense everywhere (Theorem 4) and opened (Theorem 5) on the set of DQM realizations. This dynamics has a clear structural theory. Unlike the classical systems, the attractor of DQM is defined uniquely, without alternatives (Theorem 1) and Lyapunov's exponents exist for every DQM (Theorem 3). As a Markov cascade, DQM can be with any accuracy approximated by a Markov chain, and on a compact set by a finite Markov chain (Theorem 2). This allows one to present DQM dynamics clearly and to build effective algorithms for investigating concrete systems. On the other hand, as the fluctuations tend to zero, i.e. in the semiclassical limit, the results obtained for DQM in general position become statements about initial classical dynamics. Thus, the structural stability of DQM leads to a possibility of mathematically justified numerical analysis.

This DQM method for investigating the dynamics of concrete systems and obtaining strict results is demonstrated on the example of M. Henon's system [6]. We choose the values of parameters at which this system is hyperbolical on the attractor; we will determine the support of this "strange attractor" within given error and the dynamics on it within topological equivalence. The program realized as the C # – application with usage of Open Maple technology is used here. Let us notice, however, that we get here the strict a posteriori results based on corollaries of Theorem 5 (Corollary 1 and 2) while the numerical evaluations are used only for checking the validity of assumptions of these statements.

The DQM method, as a matter of fact, is universal for investigation of systems of small dimension. The purpose of this article is only an illustration of DQM method. A detailed account of the results obtained by this method is supposed in subsequent publications.

Definition of the dynamical quantum model (DQM).

Let p(x) be an n – dimensional smooth vector field on an n – dimensional smooth Riemann manifold M, where $x(x_1, x_2, \ldots, x_n) \in R^n$ are local Euclidean coordinates on M, $p_i(x) \in C^\infty(R^n)$ (i = 1, …, n). On every phase curve x(t) $\in$ M of the dynamical system (DS), generated by this vector field,

$$\frac{dx_i}{dt} = p_i(x) \qquad (i = 1, \ldots, n), \tag{1}$$

consider the integral of the "shorten action" $s(t) = \int_{x(t)} p(x)dx = \int_0^t \|p(\tau)\|^2 d\tau$, where $\|p(\tau)\|^2 = \sum_{i=1}^n p_i^2(\tau)$. The value s(t) on each curve x(t), different from a stationary point, is diffeomorphically expressed through t and is called "optical time ". Let ρ be a metrics such that

$s(t) = \int_{x(t)} d\rho : \; d\rho = \|p(t)\|^2 dt$.

We will now give a heuristic derivation of definition of dynamical quantum model (DQM) of DS(1) (Definition 1). The distance d, covered by a point on a trajectory in time Δt is equal to

$d = \int_0^{\Delta t} \|p(\tau)\| d\tau = \|p(t_c)\| \cdot \Delta t$, where $p_c = p(t_0)$ is the average value $(0 \le t_0 \le \Delta t)$. (Single trajectory traversal is assumed in time Δt; turning points is a special case). Further, we assume that the fluctuations generate a "white noise" ξ (t) defined on the configuration space with dispersion Dξ(t) = σ^2t, where the diffusion factor σ^2 is a constant on the considered time interval. A certain time Δt should pass while the point will displace on such distance d from a starting position which will exceed the root-mean-square mistake caused ξ (t) in time Δt, i.e. $\|p_c\|\Delta t$ will exceed $\sqrt{\sigma^2 \Delta t}$. At such minimal Δt $\|p_c\|\Delta t = \sigma\sqrt{\Delta t}$, whence $\sigma^2 = \|p_c\|^2 \Delta t$ and, hence,

$$\Delta t = \frac{\sigma^2}{\|p_c\|^2}, \qquad d = \|p_c\|\Delta t = \frac{\sigma^2}{\|p_c\|} \tag{2}$$

Here, under the assumption, Δt is that minimal time interval, after which there is an opportunity to make new measurement, which difference from former will exceed an error. Thus only through time Δt we can get new significantly different measurement. Owing to (2) $\sigma^2 = \|p_c\|^2 \Delta t \approx \int_0^{\Delta t} \|p(\tau)\|^2 d\tau = s(\Delta t)$. Thus, 1) the time interval between the nearest significant measurements is constant everywhere on the scale of optical time and is equal to σ^2. (In other words the distance between them under the metrics ρ is equal to σ^2).

2) In this time " white noise " ξ (t) generates an ineradicable casual error which root-mean-square deviation is equal to the distance d on a trajectory between the nearest significant measurements.

So, the dynamic quantum model at first shifts each point on phase curve of the given dynamic system in optical time σ^2 (or on ρ – distance σ^2). And then it displaces this point casually on the distance, which is not less than the length of a trajectory from initial up to a new point. The following strict definition generalizes this description. Definition of quantum model is given for any dynamical system (1) on any compact Riemann manifold M.

Let G be a shift map on phase trajectories of dynamical system (1) in a given time Δt. We shall consider continuous function q(y, z) ≥ 0 (y, z∈M), and

$$q(y, z) > 0 \Leftrightarrow \|z - Gy\| \le d(y), \quad \int_M q(y,z)dz = 1, \quad \int_M zq(y,z)dz = Gy, \tag{3}$$

where d(y) > 0 is continuous function on M. Here q(y, z) is a density of "the local casual dissipation caused by white noise", numbers d(y) are assumed small enough. Certainly the function q(y, z) can be assumed as continuous at any given accuracy by its approximation on M by smooth function. Then

Definition 1. The Markov process with transitive function

$$P(y, A) = \int_A q(y,z)dz \ (A \subset M)$$

we will call the Dynamical quantum model (DQM) of the given dynamical system (1).

For given initial distribution we will obtain the Markov process P with this initial distribution and transitive function P(y, A). If μ_t is a distribution at time t and Δt is a time period between two nearest significant measurements, then DQM get new distribution $P(\mu_t) = \mu_{t+\Delta t}$ at time t + Δt.

Definition 2. Let Δ_i be cell of some splitting of phase space for the given dynamic system on cells in diameter ε. Let μ_0 be an initial state (initial distribution). Then the Markov chain with initial values $p_i = \mu_0(\Delta_i)$ and with probabilities p_{ij} of transition from Δ_i in Δ_j

$p_{ij} = \frac{1}{\mu_0(\Delta_i)} \int_{y\in\Delta_i} P(y,\Delta_j)\, d\mu_0$ we will call a ε – discretization of DQM with transitive function P and the initial state μ_0.

Thus, starting from the differential equation (1), we come to the difference equation with a time period at least σ^2 on a scale of optical time. At first sight step-type behavior of time in DQM can surprise: in traditional model of quantum mechanics only spatial variables errors are taken into account. But, apparently from the derivation of DQM definition, step-type behavior of process of time measurement is an inevitable consequence from presence of casual errors at coordinates and impulses. Really, a clock or some other device finally is necessary for measurement of time. But as these measure indications and speeds of their changes are determined inexactly, then also time is known with only some error [7].

DQM attractors.

Attractor is the key concept of the theory of dynamical systems. It is a “space of the established modes” on its physical sense. The point of phase space contained in attractor if it belongs to the support of “a stationary state of system”, i.e. belongs to a support of measure, which does not vary in due course.

Let phase space M is compact; P is a set DQM on M.

Definition 3. We will call a probability measure μ on M a stationary (equilibrium) state of DQM if $P\mu = \mu$. We will call the union of supports of all DQM stationary states a DQM attractor.

Theorem 1. (The Perron – Frobenius theorem for DQM). Let $\Lambda \subseteq M$ be an invariant set of DQM P, which does not contain nonempty proper invariant subsets (i.e. it is minimal with respect to P). Then 1) there is a unique stationary set μ, whose support is Λ. The state μ is ergodic.

2) For any DQM state (probability measure) ν on Λ, $\lim_{n\to\infty}\sum_{k=1}^{n} P^n\nu = \mu$.

3) If $\bar{\mu}_\varepsilon$ is a probability stationary measure of some ε – discretization of the given DQM, then $\lim_{\varepsilon\to 0}\bar{\mu}_\varepsilon = \mu$.

Proof. Let $\Lambda \subseteq M$ be the invariant closed set of DQM which is not containing proper invariant closed subsets. Let $\bar{\mu}_\varepsilon$ is a stationary measure of some discretization of given DQM on Λ with cells in diameter ε (i.e. a probability invariant measure of the Markov chain given by definition 3). A set of probability measures $M = M(\Lambda)$ is a convex metric compact set on a compact subset Λ of a phase space in weak topology [7]. Therefore from any sequence of measures $\bar{\mu}_{\varepsilon_k}$ it is possible to select subsequence $\bar{\mu}_{\varepsilon_n}$, converging to some measure $\bar{\mu}$ from M: $\lim_{n\to\infty}\bar{\mu}_{\varepsilon_n} = \bar{\mu} \in M$ in sense of weak topology on M. As $P\bar{\mu}_{\varepsilon_n} - \bar{\mu}_{\varepsilon_n} \xrightarrow[\varepsilon_n\to 0]{} 0$ (in sense of weak topology) then, owing to definition of discretization 2, $P\bar{\mu} = \bar{\mu}$, i.e. $\bar{\mu}$ is a stable DQM state. As, on the assumption, Λ does not contain nonempty proper invariant of DQM subsets (i.e. it is metrically transitive), then for any P – invariant measure $\bar{\mu}$ on Λ the ergodic theorem of Neumann is fulfilled: for any continuous function f on Λ

$$L^2 - \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(P^k) = \int f d\bar{\mu} \;. \tag{4}$$

As the left part of this equality does not depend on a choice of sequence of measures $\bar{\mu}_{\varepsilon_k}$, then any weakly converging sequence $\bar{\mu}_{\varepsilon_n}$ will converge to the same measure $\bar{\mu}$. Hence $\lim_{\varepsilon\to 0}\bar{\mu}_\varepsilon = \bar{\mu}$, it proves 3). As (4) is fulfilled for any stable state on Λ, then from (4) follows as well uniqueness of invariant measure $\bar{\mu}$, that establishes 1). At last, as for any other probability measure ν on Λ $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}P^k\nu$ exists owing to (4) and it is an invariant measure, then it is equal to $\bar{\mu}$ according to 2), as it is required.

Obviously, there is only a finite number of components of DQM attractor Λ_k on M, i.e. the invariant subsets of an attractor, which are not containing proper invariant of nonempty subsets. Any stable state on M is a convex combination of the stationary conditions μ_k on Λ_k. The following statement specifies the theorem 1, gives constructive estimations for convergence of DQM ε – discretization s to DQM.

Theorem 2. For all small enough ε, the DQM distribution μ_t at time t and the distribution of its ε – discretization μ_t^{ε} at the same time differ from each other only by a value of order $\sqrt{\varepsilon}$: $\left|\varphi(\mu_t)-\varphi(\mu_t^{\varepsilon})\right| < \mathrm{A}\|\varphi\|_0\sqrt{\varepsilon}$ $(0\le t<\infty)$, where φ is an arbitrary continuous function on M, $\|\varphi\|_0$ is its C^0 – norm, A is a constant. In particular, if $\bar{\mu}$ is a stable DQM state, $\bar{\mu}_{\varepsilon}$ is a stable state of its ε – discretization , then $\left|\varphi(\bar{\mu})-\varphi(\bar{\mu}_{\varepsilon})\right| < \mathrm{A}\|\varphi\|_0\sqrt{\varepsilon}$.

Proof. By definition, a distribution of any ε – discretization $\tilde{P}$ of DQM P on the compact manifold M will be concentrated in ε – neighborhood of DQM attractor after a finite number of steps. Therefore it is enough to obtain this theorem on a component Λ ⊆ M of DQM attractor, i.e. on the invariant closed set, which is not containing proper invariant closed subsets. It is possible to suppose without loss of generality that P (Λ) ⊆ Λ, otherwise considering P^k instead of P for such natural k that $P(\Lambda)\subseteq\Lambda_1$, $P(\Lambda_1)\subseteq\Lambda_2$, … , $P(\Lambda_{k-1})\subseteq\Lambda$.

By DQM definition, for any initial measure μ_0 on Λ distribution P(μ0) has a density function

p(y) = $\int_{y\in\Lambda} q(y,z)d\mu_0(z)$ and, thus, a density function exists for iteration $P^n(\mu_0)$ at all n > 0. Let Δ is a domain inside Λ, separated from boundary Λ on distance $\sqrt{\varepsilon}$. Let's show, that at some natural N density function $P^N(\mu_0)$ for any probability measure μ_0 on Λ in each point $z\in\Delta$ is not less, than r at some r > 0. Really, as Λ does not contain proper invariant with respect to DQM dynamics closed subsets, then any point from Λ will appear in the image of each point $y\in\Lambda$ after some number of n_y iterations of DQM. Thus, if the density function μ_0 is positive in a point y, then density function $P^{n_y}(\mu_0)$ will be positive in all interior points Λ. But n_y < N for some natural N at all $z\in\Lambda$. Really, for each point $y\in\Lambda$ there is an open set of the points passing in y as a result of one iteration of DQM. Union of these sets covers a compact set Λ. Hence it is possible to discover a final subcovering, and then N = $\max_i n_i + 1$ at chosen y_i. As function q(y, z) ≥ 0 is defined on compact set M × M by DQM definition, then q(y, z) is limited and continuous in this compact. Therefore for all probability measures μ_0 on Λ density functions $p(z) = \int_{y\in\Lambda} q(y,z)d\mu_0(z)$ are regular limited and equicontinuous. Hence the set of these functions is a precompact set in the set of continuous functions on M according to the Arzela – Ascoli theorem. And it is a compact set in view of compactness of set of measures μ_0 on Λ in weak topology. So, if densities of distributions $P^N(\mu_0)$ are close enough to zero on Δ, then some of such densities will be equal to zero on Δ, that contradicts positivity of the densities of distributions $P^N(\mu_0)$ in all interior points Λ. Thus, density function $P^N(\mu_0)\ge$ r on Δ at some r > 0 for any probability measure μ_0 on Λ. Then it is obvious that density function $P^N(\mu_0)\ge$ wr on Δ for any measure μ_0 on Λ with $\mu_0(\Lambda)$ = w <1:

$$\mu_0(\Lambda) = w < 1 \Rightarrow P^N(\mu_0)(y) \ge wr \quad (y\in\Delta) \qquad (5).$$

There is a unique probability invariant measure v: $P(v) = v$ on Λ under the theorem 1, whence $P^N(v) = v$. Then v has on Λ a nonnegative density as shown above and this density $\hat{v}(y)\ge$ r at all $y\in\Delta$ in view of (5).

Let c = max{$\hat{v}$(y)} (y ∈ Δ), f(y) = $\frac{r\hat{v}(y)}{c}$. Function $\hat{f}(y) = f(y)\big|_{\Delta}$ is f(y), limited on Δ, i.e. redefined by zero at y ∈ Λ \ Δ. Function $\tilde{f}(y) = f(y)\big|_{\Lambda\setminus\Delta}$ is f(y), limited on Λ \ Δ, i.e. redefined by zero at y ∈ Δ. Let g (y) be a density function for the measure P^N (μ_0) given by an initial probability μ_0 on Λ; let φ_1(y) = $g(y) - \hat{f}(y)$. Then φ_1 (y) ≥ 0 at all y ∈ Λ owing to (5) and $g(y) = \hat{f}(y) + \varphi_1(y) = f(y) - \tilde{f}(y) + \varphi_1(y)$. Supposing q = $\int_\Lambda \hat{f}(y)dy$, we will get that $\int_\Lambda \varphi_1(y)dy = \int_\Lambda g(y)dy - \int_\Lambda \hat{f}(y)dy$ = 1 – q. Further in this proof for arbitrary distribution μ with a density function χ (y) we will designate density function of Pμ through Pχ (y).

In such denominations $P^N\varphi_1(y) = P^N g(y) - P^N \hat{f}(y)$. But in view of (5) $P^N\varphi_1(y) \geq (1-q)\cdot r$ on Δ. Then $\varphi_2(y) = P^N\varphi_1(y) - (1-q)\cdot\hat{f}(y) \geq 0$ at all y ∈ Λ and $\varphi_2(y) = (P^N g(y) - P^N\hat{f}(y)) - (1-q)\cdot\hat{f}(y) = P^N g(y) - (P^N f(y) - P^N\tilde{f}(y)) - (1-q)\cdot(f(y) - \tilde{f}(y)) = P^N g(y) - (1 + (1-q)) f(y) + (P^N\tilde{f}(y) + (1-q)\cdot\tilde{f}(y))$, whence

$$P^N g(y) = (1 + (1-q))\, f(y) - (P^N\tilde{f}(y) + (1-q)\cdot\tilde{f}(y)) + \varphi_2(y).$$

As $\int_\Lambda \hat{f}(y)dy$ = q, and $\int_\Lambda P^N\varphi_1(y)dy = \int_\Lambda P^{N-1}\varphi_1(y)dy = \ldots = \int_\Lambda \varphi_1(y)dy = 1-q$ in accordance with DQM definition (3), then

$$\int_\Lambda \varphi_2(y)dy = \int_\Lambda P^N\varphi_1(y)dy - (1-q)\cdot\int_\Lambda \hat{f}(y)dy = 1-q-(1-q)\cdot q = (1-q)^2.$$

It is similar further $P^N\varphi_2(y) = P^{2N}g(y) - (1-q)\cdot P^N\hat{f}(y)$. As well as above, owing to (5), $P^N\varphi_2(y) \geq (1-q)^2\cdot r$ on Δ. Then $\varphi_3(y) = P^N\varphi_2(y) - (1-q)^2\cdot\hat{f}(y) \geq 0$ at all y ∈ Λ and $\varphi_3(y) = P^{2N}g_1(y) - P^{2N}\hat{f}(y) - (1-q)\cdot P^N\hat{f}(y) - (1-q)^2\cdot\hat{f}(y) = P^{2N}g(y) - (P^{2N}f(y) - P^{2N}\tilde{f}(y)) - (1-q)\cdot(P^N f(y) - P^N\tilde{f}(y)) - (1-q)^2\cdot(f(y) - \tilde{f}(y)) = P^{2N}g(y) - (1 + (1-q) + (1-q)^2) f(y) + (P^{2N}\tilde{f}(y) + (1-q)\cdot P^N\tilde{f}(y) + (1-q)^2\cdot\tilde{f}(y))$, whence $P^{2N}g(y) = (1 + (1-q) + (1-q)^2) f(y) - (P^{2N}\tilde{f}(y) + (1-q)\cdot P^N\tilde{f}(y) + (1-q)^2\cdot\tilde{f}(y)) + \varphi_3(y)$. As well as above, $\int_\Lambda \varphi_3(y)dy = \int_\Lambda \varphi_2(y)dy - (1-q)^2\cdot\int_\Lambda \hat{f}(y)dy = (1-q)^2 - (1-q)^2\cdot q = (1-q)^3$. Generally on an induction $P^{(k-1)N}\varphi_k(y) \geq (1-q)^k\cdot r$ on Δ, whence owing to (5) $\varphi_{k+1}(y) = P^N\varphi_k(y) - (1-q)^k\cdot\hat{f}(y) \geq 0$ at all y ∈ Λ and then $P^{kN}g(y) = (1 + (1-q) + (1-q)^2 + \ldots + (1-q)^k) f(y) - (P^{kN}\tilde{f}(y) + (1-q)\cdot P^{(k-1)N}\tilde{f}(y) +$

$$(1-q)^2\cdot P^{(k-2)N}\tilde{f}(y) + \ldots + (1-q)^k\tilde{f}(y)) + \varphi_{k+1}(y) \qquad (6).$$

As well as above, $\int_\Lambda \varphi_{k+1}(y)dy = \int_\Lambda \varphi_k(y)dy - (1-q)^k\cdot\int_\Lambda \hat{f}(y)dy = (1-q)^k - (1-q)^k\cdot q = (1-q)^{k+1}$, i.e. $\varphi_k(y) \underset{k\to\infty}{\to} 0$. Besides, $(1 + (1-q) + (1-q)^2 + \ldots (1-q)^k) f(y) \leq (1 + (1-q) + (1-q)^2 + \ldots (1-q)^k + \ldots) f(y) = \frac{1}{1-(1-q)} f(y) = f(y)/q = (\hat{f}(y) + \tilde{f}(y))/q$. As $\int_\Lambda \hat{f}(y)dy$ = q, then $\hat{f}(y)/q = \tilde{v}(y)\big|_\Delta$ is a unique invariant probability measure v on Λ, limited on Δ. And $\int_\Lambda (P^{kN}\tilde{f}(y) + (1-q)\cdot P^{(k-1)N}\tilde{f}(y) + (1-q)^2\cdot P^{(k-2)N}\tilde{f}(y) + \ldots + (1-q)^k\tilde{f}(y))\, dy =$

$\int_\Lambda (P^{kN}\tilde f(y))\,dy$ + (1 – q) · $\int_\Lambda (P^{(k-1)N}\tilde f(y))\,dy$ + (1 – q)^2 · $\int_\Lambda (P^{(k-2)N}\tilde f(y)\,dy$ + … + (1 – q)^k $\int_\Lambda (\tilde f(y))\,dy$ = (1 + (1 – q) + (1 – q)^2 + … (1 – q)^k) · $\int_\Lambda (\tilde f(y))\,dy \le$ (1 + (1 – q) + (1 – q)^2 + … + (1 – q)^k + …) · $\int_\Lambda (\tilde f(y))\,dy = \frac{1}{1-(1-q)} \cdot \int_\Lambda (\tilde f(y))\,dy = \frac{1}{q}\int_\Lambda (\tilde f(y))\,dy$. But volume Λ \ Δ V(Λ \ Δ) ≤ A$\sqrt{\varepsilon}$, and as f (y) = 0 on boundary Λ, at enough small ε $\tilde f(y) \le$ B$\sqrt{\varepsilon}$, B is a constant defined by function q(y, z) from DQM definition (1). Therefore $\int_\Lambda (\tilde f(y))\,dy \le$ A$\sqrt{\varepsilon}$ · B$\sqrt{\varepsilon}$ = Dε . On the other hand, at small enough ε c = max{$\hat\nu$(y)} (y ∈ Δ) is reached inside Δ; and r is the minimum of density function $P^N(\mu_0)$ on Δ is reached near to boundary Δ. Thus b$\sqrt{\varepsilon}$ ≤ r ≤ B$\sqrt{\varepsilon}$, where b, B are DQM constants. Therefore

$$q = \int_\Lambda \hat f(y)dy = \frac{r}{c}\int_\Delta \tilde\nu(y)dy \ge \frac{b}{2c}\sqrt{\varepsilon} \qquad (7),$$

as $\int_\Delta \tilde\nu(y)dy \ge \frac{1}{2}$ at enough small ε . Thus $\frac{1}{q}\int_\Lambda (\tilde f(y))\,dy \le$ Dε / $\frac{b}{2c}\sqrt{\varepsilon} = \frac{2cD}{b}\sqrt{\varepsilon}$ = C$\sqrt{\varepsilon}$.

Thereby, $P^n g_1(y)$ converges to $\tilde\nu(y)\big|_\Delta$ at n → ∞ and then at ε → 0 converges to $\tilde\nu(y)$, that establishes again 2) from the theorem 1.

In addition ε – discretization brings in described above process a discretization error, i.e. modification of order ε in distribution. Under the influence of DQM this modification is transformed to modification of order ε in density functions. Namely, for any probability distribution μ of DQM with a density function χ (y) and for its ε – discretization s $\hat\mu_\varepsilon$ with values χ_i on a cell Δ_i in diameter ε an error $d = \left|\int_M \chi(y)dy - \sum_i \chi_i\Delta_i\right| \le \sum_i \left|\int_{\Delta_i} (\chi(y)-\chi_i)dy\right| \le \sum_i \left|\max_{\Delta_i}\chi(y) - \min_{\Delta_i}\chi(y)\right| \cdot \mu(\Delta_i) \le$ bε$\sum_i \mu(\Delta_i)$ = bε, where b is DQM constant, defined by function q (y, z) from DQM definition. Thus, an estimation of an error of discretization for N steps is equal Nbε = Bε. By definition of DQM discretization $\tilde P$, dynamics of this Markov chain can be presented as at first DQM P operation, and then an average on cells Δ_i: $\tilde P\,\mu_\varepsilon = P\mu_\varepsilon + d$ and $|\varphi(\tilde P\mu_\varepsilon) - \varphi(P\mu_\varepsilon)| \le \|\varphi\|_0$bε for arbitrary continuous function φ on M, $\|\varphi\|_0$ – its C^0 – norm, whence $|\varphi(\tilde P^N\mu_\varepsilon) - P^N\mu_\varepsilon)| \le \|\varphi\|_0$Bε. On the second step $\tilde P^{2N}\,\hat\mu_\varepsilon = P^N(P^N\mu_\varepsilon + d) + d_1$, on the next step $\tilde P^{3N}\,\hat\mu_\varepsilon = P^N(P^N(P^N\mu_\varepsilon + d) + d_1) + d_2$. Generally $\tilde P^{kN}\,\mu_\varepsilon = P^N(\ldots + (P^N(P^N(P^N\mu_\varepsilon + d) + d_1) + d_2) + \ldots + d_{k-2}) + d_{k-1}$. According to process (6) $|\varphi(P^N(P^N\mu_\varepsilon + d) - P^{2N}\mu_\varepsilon)| \le$ (1 – q) · $\|\varphi\|_0$Bε, on the second step

$$|\varphi(P^N(P^N(P^N\mu_\varepsilon + d) + d_1)) - P^{3N}\mu_\varepsilon)| \le ((1-q)^2 + (1-q)) \cdot \|\varphi\|_0 B\varepsilon.$$

Generally on continuous function φ the difference between $\tilde P^{kN}\mu_\varepsilon$ and $P^{kN}\mu_\varepsilon$ is estimated from above by value (1 + (1 – q) + (1 – q)^2 + … + (1 – q)^k + …)Bε = Bε/q. As q ≥ $\frac{b}{2c}\sqrt{\varepsilon}$ by (7), then the upper bound is equal Bε/$\frac{b}{2c}\sqrt{\varepsilon} = \frac{2Bc}{b}\sqrt{\varepsilon}$ = E$\sqrt{\varepsilon}$, as it is required.

There is a unique probability invariant measure μ on a component Λ of DQM attractor, $P(\mu) = \mu$ under the theorem 1. Hence $P^N(\mu) = \mu$. On proved μ has a nonnegative density function on

Λ and positive on Δ. On trajectory space of DQM, as on casual process, the invariant measure $\bar{\mu}$, induced there by a measure μ, is defined according to Kolmogorov's theorem [9]. Further the product $DG(y_n) \cdot \ldots \cdot DG(y_1) \cdot DG(y_0)$, where $y_0, y_1, \ldots, y_n$ is a DQM trajectory ω at time t_n and $DG(y_k)$ is a differential of initial smooth DS in a point $y_k \in \Lambda$, we will call the DQM differential $D_n(\omega)$ for DQM trajectory ω at time t_n.

Theorem 3. Let Λ be an invariant set of DQM of dimension m not containing proper invariant nonempty subsets. Then we have for DQM with small enough $d(y)$ that

1) For almost all with respect to measure $\bar{\mu}$ DQM trajectories ω and for every nonzero vector $u \in \mathbf{R}^m$ the limits exist

$$\lim_{n\to\pm\infty} \frac{1}{|n|} \ln\|D_n(\omega)u\| = \pm\lambda_r,$$

where $\lambda_r = \lambda_r(u)$, $r \le m$.

2) For every point y and for every such trajectory ω, the filtrations of subspaces are uniquely defined:

forward $L^+_1(y) \subset L^+_2(y) \subset \ldots \subset L^+_s(y) = \mathbf{R}^m$ and backward $L^-_s(y) \subset \ldots \subset L^-_2(y) \subset L^-_1(y) = \mathbf{R}^m$,

connected with the numbers λ_r in such a way that $\lambda_1 < \lambda_2 < \ldots < \lambda_s$ and

$$\lim_{n\to\infty} \frac{1}{n} \ln\|D_n(\omega)u\| = \lambda_r \Leftrightarrow u \in L^+_r(y) \text{ и } u \notin L^+_{r-1}(y),\quad \lim_{n\to-\infty} \frac{1}{n} \ln\|D_n(\omega)u\| = \lambda_r \Leftrightarrow u \in L^-_r(y)$$

и $u \notin L^-_{r+1}(y)$.

These filtrations are invariant with respect to DQM differential. Namely, if y_n and y_{n+1} are consecutive points of a trajectory ω at the moments t_n and t_{n+1}, then the differential $D_n(\omega)$ translates the filtrations in a point y_n into a filtration in a point y_{n+1}.

By analogy to the theory of smooth dynamic systems, we will name the numbers λ_r as Lyapunov's characteristic exponents for a component Λ of DQM attractor.

Proof. We will consider DQM on Λ, as casual process $X(t, \omega)$, where t is discrete time, $t = t_k$, $k = 0, 1, 2, \ldots$, ω – DQM trajectory. Namely, let $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_k, \ldots)$, where $\eta_k \in M$. Then trajectory of DQM $\omega = \omega(t, y_0)$ with an initial point y_0 is sequence $X(t_0, \omega) = y_0$, $X(t_1, \omega) = y_1 = Gy_0 + \eta_1$, $X(t_2, \omega) = y_2 = Gy_1 + \eta_2$, …, $X(t_k, \omega) = y_k = Gu_{k-1} + \eta_k$, …. (Here all $d(y)$ are assumed so small that addition $Gu_{k-1} + \eta_k$, where $\|\eta_k\| \le d(y_k)$, is fulfilled on a local map of manifold M in R^n). Thus, DQM trajectory ω is given univalently by sequence of vectors $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_k, \ldots)$ and an initial point y_0: $\omega = \omega(y_0; \boldsymbol{\eta})$.

On space Ω of the DQM trajectories $X(t, \omega)$ on Λ DQM induces the dynamic process T: $T\omega_0 = \omega_1$, $T\omega_1 = \omega_2$, …, $T\omega_{k-1} = \omega_k$, …., where $\omega_0 = \omega(y_0; \boldsymbol{\eta_0})$, $y_0 = X(t_0, \omega)$, $\boldsymbol{\eta_0} = (\eta_1, \eta_2, \ldots, \eta_m, \ldots)$, and $\omega_1 = \omega(y_1; \boldsymbol{\eta_1})$, $y_1 = Gy_0 + \eta_1 = X(t_1, \omega)$, $\boldsymbol{\eta_1} = (\eta_2, \eta_3, \ldots, \eta_m, \ldots)$. Thus T is a trajectory of trajectories. If $\omega_2 = (y_2; \boldsymbol{\eta_2})$, then $y_2 = Gy_1 + \eta_2 = X(t_2, \omega)$, $\boldsymbol{\eta_2} = (\eta_3, \eta_4, \ldots, \eta_m, \ldots)$. Generally for $\omega_k = (y_k; \boldsymbol{\eta_k})$ we will obtain $y_k = Gu_{k-1} + \eta_k = X(t_k, \omega)$, $\boldsymbol{\eta_k} = (\eta_{k+1}, \eta_{k+2}, \ldots, \eta_m, \ldots)$.

According to Kolmogorov's theorem [9] on a trajectory space Ω the probability invariant measure $\bar{\mu}$, induced there by a stable state μ on Λ, is defined. T is an endomorphism of Ω, keeping measure $\bar{\mu}$ and ergodic on construction owing to ergodicity of DQM on Λ. Let's suppose $a(n, \omega) = D_n(\omega)$ for $\omega = \omega(y; \boldsymbol{\eta})$ (here $y = y_0$). Then $a(n + k, \omega) = a(k, T^n\omega) \cdot a(n, \omega)$. This equality means that square matrixes $a(n, \omega)$ of an order m are a multiplicative cocycle on a trajectory space Ω with respect to its endomorphism T.

As maps $G(y)$ are diffeomorphisms, then $\|DG(y)\| \ne 0$ at all $y \in \Lambda$. Thus $\ln\|DG(y)\|$ is a continuous function on a compact set Λ and $\int_{y\in\Lambda} \ln\|DG(y)\| d\mu < \infty$. But for any characteristic function χ_C of an open subset $C \subseteq \Lambda$ $\int_{\Omega} \chi_C \, d\bar{\mu} = \bar{\mu}(\{\omega = (y; \boldsymbol{\eta}) \mid y \in C\}) = \mu(\{y \mid y \in C\}) =$

= $\int\limits_M \chi_C \, d\mu$ by measure $\overline{\mu}$ definition. Therefore for any sectionally continuous function g on M $\int\limits_\Omega g\, d\overline{\mu} = \int\limits_M g\, d\mu$. As on each trajectory ω = ω(y; **η**) a(0, ω) = D_0 (y), then $\int\limits_{\omega\in\Omega} \ln\|a(0,\omega)\| d\overline{\mu}$ = $\int\limits_{y\in\Lambda} \ln\|DG(y)\| d\mu$ < ∞. This inequality means that for a cocycle a(n, ω) the multiplicative ergodic theorem [10] is fulfilled.

This theorem states that almost all DQM trajectories ω ∈ Ω with respect to measure $\overline{\mu}$ are correct on Lyapunov. It means, in particular, that

1) for such ω there are limits

$$\lim_{n\to\pm\infty} \frac{1}{|n|} \ln\|a(n,\omega)u\| = \pm\, \lambda_r(\omega),$$

where r = 1, 2, … s = s (ω), s (ω) ≤ m.

2) On each such trajectory ω filtrations of subspaces are univalently defined: forward $L^+_1(\omega) \subset L^+_2(\omega) \subset \ldots \subset L^+_s(\omega) = \mathbf{R}^m$ and backward $L^-_s(\omega) \subset \ldots \subset L^-_2(\omega) \subset L^-_1(\omega) = \mathbf{R}^m$, connected with numbers $\lambda_1(\omega) < \lambda_2(\omega) < \ldots < \lambda_s(\omega)$ (s = s (ω)) such that

$$\lim_{n\to\infty} \frac{1}{n} \ln\|a(n,\omega)u\| = \lambda_r(\omega) \quad \Leftrightarrow \quad u \in L^+_r(\omega) \quad и \quad u \notin L^+_{r-1}(\omega),$$

$$\lim_{n\to-\infty} \frac{1}{n} \ln\|a(n,\omega)u\| = \lambda_r(\omega) \quad \Leftrightarrow \quad u \in L^-_r(\omega) \quad и \quad u \notin L^-_{r+1}(\omega).$$

These filtrations are invariant with respect to endomorphism T. Namely, if T (ω_n) = ω_{n+1} then cocycle a(n, ω) translates filtration ω_n in a filtration ω_{n+1}. As T is ergodic on Ω, then $\lambda_r(\omega)$ does not depend from ω, i.e. $\lambda_r(\omega) \equiv \lambda_r$ almost everywhere with respect to measure $\overline{\mu}$; similarly almost everywhere s (ω) ≡ s. In view of correspondences a(n, ω) = D_n (ω), ω(*y*; **η**) → *y*, from here the theorem statement directly follows, as it is required.

Thus DQM attractor is defined uniquely, without alternatives. It can be investigated algorithmically as DQM, the Markov cascade, and it can be approximated by a Markov chain. Thus, received discrete dynamics has the clear structural theory and good algorithms of research of concrete systems, and at $\varepsilon \to 0$ passes in DQM.

Structural stability in DQM.

Let η = η (t, y) ∈ M is a smooth vector field on phase manifold M, where y ∈ M, t is a discrete time: t = t_k, k = 0, 1, 2, …, $\|\eta(t,y)\|_{C^n} \le d(y)$. On sense η is a small casual deviation, called by white noise, in a point y at time t. Then at the set field η(t, y) a trajectory ω of DQM, as casual process X(t, ω): X(t_0, ω) = y_0, X(t_1, ω) = y_1 = Gy_0 + η(t_1, y_0), …, X(t_k, ω) = y_k = Gu_{k-1} + η(t_k, u_{k-1}), … is uniquely given by an initial point y_0 ∈ M. (All d (y) are assumed small enough that addition Gu_{k-1} + η(t_k, u_{k-1}) is fulfilled on a local map of manifold M in R^n).

Definition 4. Let η(t, y) ∈ M is a given smooth field on R × M, $\|\eta(t,y)\|_{C^n} \le d(y)$. Then the sequence of maps G_1(y) = G(y) + η (t_1, y),…, G_k(y) = G(y) + η(t_k, y), … we will call a DQM realization.

All maps G_k (y) are diffeomorphisms on M at enough small d(y). DQM is a smooth stratification over base of the M, which stratum is the set of sequences of vectors **η**(y) = (η(t_1, y),…, η(t_k, y), …) at all admissible smooth fields of vectors η(t, y). Then DQM realizations are cuts of such smooth stratification. It is possible to set DQM realization on subset S of manifold M, invariant with respect to this realization, and thus considering it, as a base of sub stratification. It is natural to consider generalization DQM, constructed not on map of shift G, but on some realization.

Let G_k (y) (k = 0, 1, 2, …) is some DQM realization; q_k(y, z) ≥ 0 (y, z ∈ M) is such smooth functions regular limited in C^1, that at any k

$$q_k(y, z) > 0 \Leftrightarrow \|z - G_k(y)\| \le d(y), \quad \int_M q_k(y,z)dz = 1, \quad \int_M z_k q(y,z)dz = G_k(y), \tag{8}$$

where d (y)> 0 is a continuous function on M. Then

Definition 1 `. Dynamic quantum model for given realization $G_k(y)$ (k = 0, 1, 2, …) we will call nonautonomous Markov process with a transition function

$$P_k(y, A) = \int_A q_k(y,z)dz \ \ (A \subset M).$$

For DQM in sense of definition 1` at the set smooth vector field η(t, y) on phase manifold M its trajectory ω, as casual process X(t, ω), is the sequence $X(t_0, \omega) = y_0$, $X(t_1, \omega) = y_1 = G_1 y_0 + \eta_1$, $X(t_2, \omega) = y_2 = G_2 y_1 + \eta_2$, …, $X(t_k, \omega) = y_k = G_k u_{k-1} + \eta_k$, …, where all $\eta_k \in M$. The statement and the proof of the theorem 3 literally are transferred on such DQM. Further in all statements DQM is understood in sense of definition 1, but at proofs will be used also DQM in sense of definition 1` and it will be always specially mentioned. Following definition we will introduce by analogy to the theory of smooth dynamic systems.

Definition 5. DQM realization $G_k(y)$ (k = 0, 1, 2, …) with set of initial points on compact set K ⊆ M, invariant with respect to this realization, we will call hyperbolic DQM realization on K, if in any point y ∈ K and at any time t_k there is an expansion of tangential stratification TK in the sum of Whitney sub stratifications $E_k^s(y)$ and $E_k^u(y)$: $TK = E_k^s(y) \oplus E_k^u(y)$, satisfying to such conditions:

1) Tangential map DG_k keeps sub stratifications:

$DG_k(E^s_k) \subseteq E^s_{k+1}$, $DG_k(E^u_k) \subseteq E^u_{k+1}$;

2) DG_k contracts E^s_k; more precisely, there will be such constants c > 0 and λ (0 <λ <1) that at any $v \in E^s_k$ and any natural n,

$$\|DG_{k+n} ... DG_k(v)\| \le c\lambda^n \|v\|;$$

3) DG_k stretches E^u_k, i.e. for any $v \in E^u_k$ and natural n with the same c and λ

$$\|DG_{k+n} ... DG_k(v)\| \ge \frac{1}{c\lambda^n}\|v\|.$$

Theorem 4. The hyperbolic realizations are everywhere dense on the set of DQM realizations. More precisely, for any DQM realization $G_k(y)$ (k = 0, 1, 2, …) with the set of initial points on a subset S ⊆ M invariant with respect to this realization, and for small enough ε > 0 there exists a hyperbolic realization $\overline{G}_k(y)$ of this DQM on compact set K ⊆ M such that 1) μ (S/K ∪ K/S) < ε for the probability invariant measure μ of this DQM; 2) on S ∩ K we have $\|G_k(y) - \overline{G}_k(y)\|_{C^1} < \varepsilon$ (k = 0, 1, 2, …).

Proof. It is enough to prove this statement assuming that S = Λ is a component of attractor of DQM (generally at the discovered compact set K for Λ taking K ∩ S for given S). We will consider DQM X(t, ω) for the dynamic system, generated by realization $G_k(y)$ (k = 0, 1, 2, …) on Λ, in which d(y) = ε/3 for all y ∈ M at small enough ε > 0. On a connective component Λ of DQM attractor the unique stable state μ is defined, owing to the theorem 1. By Kolmogorov's theorem [9], the invariant measure $\overline{\mu}$ is defined on trajectory space of DQM, induced there by a measure μ. Then for the almost all DQM trajectories X(t, ω) with respect to the measure $\overline{\mu}$ statements of the theorem 3 are fulfilled. We will show that it is possible to find such smooth realization $\widehat{G}_k(y)$ of this DQM, in which the measure of all trajectories in a general position is equal to 1, i.e. the theorem 3 is fulfilled for a full measure of initial points from Λ.

Really, we will consider smooth DQM realizations X(t, ω), consisting of such trajectories ω = ω(y; **η**), **η** = (η_1, …, η_k, …), that η_k does not depend on y∈Λ at all k = 0, 1, 2, …. Such realizations, as sets of trajectories, are not intersected at different **η**. And their union contains all trajectories ω = ω(y; **η**), coincides with all stratification, i.e. this is partition of DQM. As the measure of all trajectories in general position is equal to 1, so the measure of atypical trajectories is distinct from zero only on a zero measure of such smooth realizations. For any smooth realization $\widehat{G}_k(y)$ of this DQM, in which the measure of trajectories in general position is equal to 1, is also $\left\|G_k - \widehat{G}_k\right\|_{C^1} \leq \varepsilon/3$ by construction.

Now we will rebuild DQM X(t, ω). At first at each point *y* we will bound the random deviation, given by continuous function q(y, z) ≥ 0 in definition of DQM, to deviation radius d(y)/$e^{\varepsilon/4}$ (instead of d(y) for DQM X (t, ω)). And then we will make linear expansion (homothety) with factor of expansion $e^{\varepsilon/4}$. Then we get a DQM $\widetilde{X}(t,\omega)$, which by construction coincides pointwisely with X(t, ω) and has the same trajectories and realizations, but all Lyapunov's characteristic exponents on the component Λ increase by ε/4. If some characteristic exponent of X(t, ω) is zero, then all characteristic exponents of $\widetilde{X}(t,\omega)$ will be nonzero for small enough ε as a result of such reorganization. Thus smooth realization $\widetilde{G}_k(y)$ of DQM X(t, ω) will be transformed into a smooth realization $\widehat{G}_k(y)$ of $\widetilde{X}(t,\omega)$, coinciding pointwisely with $\widetilde{G}_k(y)$, but its Lyapunov's characteristic exponents are nonzero for almost all y ∈ Λ. And $\left\|\widetilde{G}_k - \widehat{G}_k\right\|_{C^1} \leq \varepsilon/3$ for small enough ε on construction.

We will suppose $E^s{}_0(y) = L^+{}_r(y)$ for realization $\widetilde{G}_k(y)$ with nonzero Lyapunov's characteristic exponents for any *y* on a trajectory in general position. Subspace $L^+{}_r(y)$, defined in theorem 3, is such, that λ_r- has the greatest (i.e. the least modulo) negative characteristic exponent. We will suppose further that $E^s{}_1(y) = D\widetilde{G}_0(y)(E^s{}_0(y))$, …, $E^s{}_{k+1}(y) = D\widetilde{G}_k(y)(E^s{}_k(y))$, …. We will similarly suppose $E^u{}_0(y) = L^-{}_{r+1}(y)$, where λ_{r+1} is the least positive characteristic exponent, and further $E^u{}_1(y) = D\widetilde{G}_0(y)(E^u{}_0(y))$, …, $E^u{}_{k+1}(y) = D\widetilde{G}_k(y)(E^u{}_k(y))$, …. Thus the sum of Whitney substratifications $E^s{}_k(y)$ and $E^u{}_k(y)$ of tangential stratification TK are given in any point *y* of a typical trajectory: $TK = E^s{}_k(y) \oplus E^u{}_k(y)$. This expansion is invariant with respect to differential of DQM $\widetilde{X}(t,\omega)$ by construction. Besides, $\lim\limits_{n\to\infty}\frac{1}{n}\ln\left\|D_n(\omega)u\right\| \leq \lambda_r$ for $u \in L^+{}_r(y)$ by theorem 3. Hence $\frac{1}{n}\ln\left\|D_n(\omega)u\right\| < \lambda_r/2$ on compact $\|u\| = 1$ in $L^+{}_r = E^s{}_0$, if n > N for large enough N (here $\lambda_r < 0$). Then for n > N $\left\|D_n(\omega)u\right\| < \alpha^n$, where $\alpha = e^{\lambda_r/2} < 1$, $\|u\| = 1$. So $\frac{\left\|D_n(\omega)u\right\|}{\|u\|} < \alpha^n$ at all $u \in E^s{}_0$. Let's $c_s = c_s(\omega) = \max\limits_{n\leq N}\frac{\alpha^n\|u\|}{\left\|D_n(\omega)u\right\|}$; then $\left\|D_n(\omega)u\right\| \leq c_s\alpha^n\|u\|$ for all $u \in E^s{}_0(y)$, i.e. $E^s{}_0(y)$ sets contracting foliation. Similarly, taking $\beta = e^{\lambda_{r+1}/2} > 1$, such N that for n > N $\left\|D_n(\omega)u\right\| > \beta^n\|u\|$ and $c_u = c(\omega) = \min\limits_{n\leq N}\frac{\beta^n}{\left\|D_n(\omega)u\right\|}$, we will obtain, that $\left\|D_n(\omega)u\right\| \geq c_u\beta^n\|u\|$ for all $u \in E^u{}_0(y)$, i.e. $E^u{}_0(y)$ sets stretching foliation. At last, having chosen $\lambda = \min\{\alpha; 1/\beta\}$, $c = c(\omega) = \max\{c_s, c_u\}$, on the given trajectory ω with an index point *y* we will obtain, that

$$\left\|DG_{k+n}...DG_k(v)\right\| \leq c\lambda^n\|v\| \quad (v \in E_0^s(y)), \qquad \left\|DG_{k+n}...DG_k(v)\right\| \geq \frac{1}{c\lambda^n}\|v\| \quad (v \in E_0^u(y))$$

at all n and k according to definition 5.

It is obvious that DQM trajectories ω for which *c* = c(ω) < *C* are closed, i.e. make compact set K_C. Union of compact sets K_C on all *C* > 0 coincides with the set of points of all typical trajectories of DQM X(t, ω) on construction. We will choose *C* so large that μ(S/K ∪ K/S) < ε for a given ε > 0 and K = K_C. Then inequalities

$$\left\|DG_{k+n}...DG_k(v)\right\| \leq C\lambda^n\|v\| \quad (v \in E_0^s(y)), \qquad \left\|DG_{k+n}...DG_k(v)\right\| \geq \frac{1}{C\lambda^n}\|v\| \quad (v \in E_0^u(y)),$$

are fulfilled for all $y \in$ K, as it is required.

For DQM realization it is possible to convert time: to study the sequence $(\ldots, G_{-k}(x), \ldots, G_{-1}(x), G_0(x))$ at $G_{-k}(x) = G_k(x)$ is equivalently to study realization $(G_0(x), G_1(x), \ldots, G_k(x), \ldots)$. Naturally therefore to generalize concept of realization on arbitrary two-sided sequences $G_k(x)$ $(k = 0, \pm1, \pm2, \ldots)$: $(\ldots, G_{-k(x)}, \ldots, G_{-1}(x), G_0(x), G_1(x), \ldots, G_k(x), \ldots)$.

Definition 6. DQM realization $G_k(x)$ $(k = 0, \pm1, \pm2, \ldots)$ with the set of initial points on compact set $K \subseteq M$, invariant with respect to this realization, we will call structurally stable if any realization $\tilde{G}_k(x)$ of this DQM $(k = 0, \pm1, \pm2, \ldots, x \in K)$, close enough to $G_k(x)$ in C^1 – topology, is topologically equivalent G_k, i.e. there are such homeomorphisms H_k that $\tilde{G}_k \circ H_k = H_{k+1} \circ G_k$ $(k = 0, \pm1, \pm2, \ldots)$.

In applications, as a rule, compact set K is a DQM attractor and by that is the closure of some invariant area of DQM in M.

Theorem 5. Any hyperbolic DQM realization $G_k(x)$ $(k = 0, 1, 2, \ldots)$ with the set of initial points on invariant compact set $K \subseteq M$ of this realization, is structurally stable.

Proof. By definition, it is required for any $k = 0, \pm1, \pm2, \ldots$ to find such homeomorphism $H_k(x) = x + h_k(x)$, that the following diagram is commutative.

$$\begin{tikzcd} K \arrow[r, "\tilde{G}_k"] & K \\ K \arrow[u, "H_k"] \arrow[r, "G_k"] & K \arrow[u, "H_{k+1}"'] \end{tikzcd}$$

Here $\tilde{G}_k(x) = G_k(x) + f_k(x)$ $(x \in K)$ where all functions $f_k(x) \in C^1$ also are small enough in C^1 – topology. Let DG_{kx} be the differential of $G_k(x)$ at a point x; R_{kx} is a nonlinear part of $G_k(x)$ in a point x: $G_k(x + h_k(x)) = G_k(x) + DG_{kx}(h_k(x)) + R_{kx}(h_k(x))$. From the diagram we obtain $\tilde{G}_k \circ H_k = H_{k+1} \circ G_k$ $(k = 0, \pm1, \pm2, \ldots)$, whence

$$G_k(x + h_k(x)) + f_k(x + h_k(x)) = G_k(x) + h_{k+1}(G_k(x));$$
$$G_k(x) + DG_{kx}(h_k(x)) + R_{kx}(h_k(x)) + f_k(x + h_k(x)) = G_k(x) + h_{k+1}(G_k(x));$$
$$h_{k+1}(G_k(x)) - DG_{kx}(h_k(x)) = R_{kx}(h_k(x)) + f_k(x + h_k(x)). \tag{9}$$

It is the system of nonlinear functional equations with respect to sequence of functions $h_k(x)$ $(k = 0, \pm1, \pm2, \ldots)$. We will get the corresponding homologous equations, i.e. we linearize (9):

$$h_{k+1}(G_k(x)) - DG_{kx}(h_k(x)) = f_k(x). \tag{10}$$

Let's suppose $L_k = h_{k+1}(G_k(x)) - DG_{kx}(h_k(x))$ $(k = 0, \pm1, \pm2, \ldots)$. Then $L = (\ldots, L_{-1}, L_0, L_1, \ldots)$ is a linear operator on Banach space **B** of sequences of regularly limited continuous vector – functions $\mathbf{h} = (\ldots, h_{-1}(x), h_0(x), h_1(x), \ldots)$ with norm $\| \mathbf{h} \|_B = \sup_k(\max_K |h_k(x)|)$ $(k = 0, \pm1, \pm2, \ldots)$. We will prove convertibility of operator L: then a solution of the system of homologous equations $L(\mathbf{h}) = \boldsymbol{f}$ looks like $\mathbf{h} = L^{-1}(f)$, where $\boldsymbol{f} = (\ldots, f_{-1}(x), f_0(x), f_1(x), \ldots)$. Let's assume for simplicity and without loss of generality that the Riemannian metric on M is the Lyapunov's one with respect to hyperbolic realization $G_k(x)$. Then any $\mathbf{h} \in \mathbf{B}$ for small enough $\| \mathbf{h} \|_B$ can be so spread out on contracting and stretching foliations $h_k(x) = h_{ks}(x) + h_{ku}(x)$, that for some μ $(0 < \mu < 1)$ $\|DG_{kx}(h_{ks}(x)\| < \mu \|h_{ks}(x)\|$, $\|DG_{kx}(h_{ku}(x)\| > \frac{1}{\mu}\|h_{ku}(x)\|$, where $\| \ \|$ is a C^0 – norm. Thus also (10) breaks up on two subsystems:

$L_s(\mathbf{h}_s) = \boldsymbol{f}_s$: $h_{k+1s}(G_k(x)) - DG_{kx}(h_{ks}(x)) = f_{ks}(x)$ (11s)

$L_u(\mathbf{h}_u) = \boldsymbol{f}_u$: $h_{k+1u}(G_k(x)) - DG_{kx}(h_{ku}(x)) = f_{ku}(x)$. (11u)

Let's consider operators S($\mathbf{h}$) on $\mathbf{B}$: $h_k(x) \to h_{k+1}(G_k(x))$ and DG($\mathbf{h}$): $h_k(x) \to DG_{kx}(h_k(x)$. Here $\| S \|_B = \| S^{-1} \|_B = 1$, and $\| DG \|_B \le \mu$. Then the system of the homologous equations (11s) on contracting foliation can be noted in the form $L_s(\mathbf{h}_s) = (S - DG)(\mathbf{h}_s) = = \boldsymbol{f}_s$, whence $S(E - S^{-1} \cdot DG)(\mathbf{h}_s) = \boldsymbol{f}_s$. As $\| S^{-1} \cdot DG \|_B \le \| S^{-1} \|_B \cdot \| DG \|_B \le \mu < 1$, then operator $E - S^{-1} \cdot DG$ is reversible and

$(E - S^{-1} \cdot DG)^{-1} = E + S^{-1} \cdot DG + (S^{-1} \cdot DG)^2 + (S^{-1} \cdot DG)^3 + \dots$,

$\| (E - S^{-1} \cdot DG)^{-1} \|_B \le \dfrac{1}{1-\mu}$. Thus operator L_s is reversible also: $L_s^{-1} = (E - S^{-1} \cdot DG)^{-1} \cdot S^{-1}$ and $\| L_s^{-1} \|_B \le \dfrac{1}{1-\mu}$. On stretching foliation $L_u(\mathbf{h}_u) = (S - DG)(\mathbf{h}_u) = \boldsymbol{f}_u$. Here operator DG is reversible and $\| DG^{-1} \|_B \le \mu$. Consequently operator $L_u = S - DG = (S \cdot DG^{-1} - E) \cdot DG$ also is reversible, as $\| S \cdot DG^{-1} \|_B \le \| S \|_B \cdot \| DG^{-1} \|_B \le \mu < 1$. Therefore $(E - S \cdot DG^{-1})^{-1} = E + S \cdot DG^{-1} + (S \cdot DG^{-1})^2 + (S \cdot DG^{-1})^3 + \dots$,

$\| (E - S \cdot DG^{-1})^{-1} \|_B \le \dfrac{1}{1-\mu}$ and $\| L_u^{-1} \|_B = \| DG^{-1} \cdot (S \cdot DG^{-1} - E)^{-1} \|_B \le \dfrac{\mu}{1-\mu}$. Thus, operator L is reversible and $\| L^{-1} \|_B \le \dfrac{1}{1-\mu}$.

We will solve now by a method of contracting maps the nonlinear functional equation (1) on $\mathbf{B}$. We will suppose $F(\mathbf{h}) = (\dots, R_{-1x}(h_{-1}(x)) + f_{-1}(x + h_{-1}(x)), R_{0x}(h_0(x)) + f_0(x + h_0(x)), R_{1x}(h_1(x)) + f_1(x + h_1(x)), R_{2x}(h_2(x)) + f_2(x + h_2(x)), \dots) \in \mathbf{B}$. We will give the iterative equation by the formula $\mathbf{h}_i = L^{-1}(F(\mathbf{h}_{i-1}))$. Let`s initial iteration $\mathbf{h}_0 \equiv 0$. Then the first iteration $\mathbf{h}_1 = L^{-1}(F(\mathbf{h}_0))$, whence $L(\mathbf{h}_1) = F(\mathbf{h}_0) = (\dots, f_{-1}(x), f_0(x), f_1(x), \dots) = \boldsymbol{f}$. Thus, $\mathbf{h}_1$ is a solution of system of the homologous equations (10): $\mathbf{h}_1 = L^{-1}(f)$. Generally $\mathbf{h}_{i-1} = (\dots, h_{-1i-1}(x), h_{0i-1}(x), h_{1i-1}(x), \dots) \in \mathbf{B}$ for iteration $i - 1$, $\mathbf{h}_i =$

$L^{-1}(F(\mathbf{h}_{i-1})) = (\dots, R_{-1x}(h_{-1\,i-1}(x)) + f_{-1}(x + h_{-1\,i-1}(x)), R_{0x}(h_{0\,i-1}(x)) + f_0(x + h_{0\,i-1}(x)), R_{1x}(h_{1i-1}(x)) + f_1(x + h_{1\,i-1}(x)), \dots) \in \mathbf{B}$ for iteration i. Here

$\| \mathbf{h}_1 \|_B \le \| L^{-1}(\boldsymbol{f}) \|_B \le \| L^{-1} \|_B \cdot \| \boldsymbol{f} \|_B \le \dfrac{1}{1-\mu} \cdot \| \boldsymbol{f} \|_B$, $\| \mathbf{h}_{i+1} - \mathbf{h}_i \|_B \le$

$$\left\|L^{-1}(F_i) - L^{-1}(F_{i-1})\right\|_B = \left\|L^{-1}(F_i - F_{i-1})\right\|_B \le \left\|L^{-1}\right\|_B \cdot \left\|F_i - F_{i-1}\right\|_B \le \frac{1}{1-\mu} \cdot \left\|F_i - F_{i-1}\right\|_B .$$

The component k of vector $F_i - F_{i-1} \in \mathbf{B}$ is equal to $[R_{kx}(h_{k\,i}(x)) + f_k(x + h_{k\,i}(x))] - [R_{kx}(h_{k\,i-1}(x)) + f_k(x + h_{k\,i-1}(x))] = [R_{kx}(h_{k\,i}(x)) - R_{kx}(h_{k\,i-1}(x))] + [f_k(x + h_{k\,i}(x) - f_k(x + h_{k\,i-1}(x))]$.

By definition, $\left\|R_{kx}(h_k(x)\right\| \le \left\|G_k(x)\right\|_2 \cdot \left\|h_k(x)\right\|^2$, where $\| \; \|_2$ is a norm in C^2, $\| \; \|$ is a norm in C^0.

All $G_k(x)$ differ from G(x) only on value of an order $\varepsilon = \max_{y\in K} d(y)$, therefore obviously $\left\|G_k(x)\right\|_2 < 2\left\|G(x)\right\|_2$ at enough small ε and any $k = 0, \pm1, \pm2, \dots$.

Therefore

$$\left\|R_{kx}(h_{ki}(x)) - R_{kx}(h_{k\,i-1}(x))\right\| \le 2\left\|G(x)\right\|_2 \left(\left\|h_{ki}(x)\right\| + \left\|h_{k\,i-1}(x)\right\|\right) \cdot \left\|h_{ki}(x) - h_{k\,i-1}(x)\right\| \;;$$

$$\left\|R_{kx}(h_{ki}(x)) - R_{kx}(h_{k\,i-1}(x))\right\| \le 4\left\|G(x)\right\|_2 \cdot \max\{\left\|h_{ki}(x)\right\|, \left\|h_{k\,i-1}(x)\right\|\} \left\|h_{ki}(x) - h_{k\,i-1}(x)\right\| \quad (12)$$

Let's establish parameters at which this iterative process will be contracting. Let $\alpha \le \min\{\dfrac{1-\mu}{32\left\|G(x)\right\|_2}, \dfrac{1-\mu}{32}\}$; O is union of all full-spheres of radius α with the centers in points of compact set K ($K \subseteq O \subseteq M$). Then the small enough continuous vector – function $h_k(x)$ on O can be decompose $h_k(x) = h_{ks}(x) + h_{ku}(x)$ for small enough α such that conditions $\|DG_{kx}(h_{ks}(x)\| < \mu\|h_{ks}(x)\|$, $\|DG_{kx}(h_{ku}(x)\| > \frac{1}{\mu}\|h_{ku}(x)\|$ will be continued from compact set K to a neighborhood $O \supseteq K$ by a continuity, perhaps increasing μ by value of order α (i.e. worsening an estimate). From this also inequality $\|L^{-1}\|_B < \frac{1}{1-\mu}$ will be continued from the sequences of continuous vector – functions **h** on K to sequences of vector – functions on O as proved above, perhaps with worsening the estimate by an order α. Obviously the inequality $\|L^{-1}\|_B < \frac{2}{1-\mu}$ will be fulfilled for small α for **h** on O.

We get for the first iteration $\mathbf{h}_1$, i.e. for a solution of system of the homologous equations

$$||\mathbf{h}_1||_B = ||L^{-1}(f)||_B \le \|L^{-1}\|_B \cdot \|f\|_B \le \frac{2}{1-\mu} \cdot \frac{1-\mu}{2} \cdot \frac{\alpha}{2} = \frac{\alpha}{2},$$

supposing $\|f_k\|_1 < \frac{1-\mu}{2} \cdot \frac{\alpha}{2}$ for all k = 0, ±1, ±2, …, where $\| \ \|_1$ – norm in C^1.

Let's prove by induction that from $||\mathbf{h}_i||_B \le \alpha$ at i = 1, 2, …, m follows that $||\mathbf{h}_{m+1}||_B \le \alpha$ and $\| \mathbf{h}_{i+1} - \mathbf{h}_i \|_B \le \frac{1}{2} \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B$ ($1 \le i \le m$). Really, from $\|h_{ki}(x)\| \le \alpha$, $\|h_{k\,i-1}(x)\| \le \alpha$ for all x and k by (12) follows that

$$\|R_{kx}(h_{ki}(x)) - R_{kx}(h_{k\,i-1}(x))\| \le 4\|G(x)\|_2 \cdot \alpha \cdot \|h_{ki}(x) - h_{k\,i-1}(x)\| \le$$
$$\le 4\|G(x)\|_2 \cdot \frac{1-\mu}{32\|G(x)\|_2} \|h_{ki}(x) - h_{k\,i-1}(x)\| \le \frac{1-\mu}{8} \|h_{ki}(x) - h_{k\,i-1}(x)\|.$$

On the other hand, at all k $\| f_k(x + h_{k\,i}(x) - f_k(x + h_{k\,i-1}(x)) \| \le \|f_k\|_1 \cdot \|h_{ki}(x) - h_{k\,i-1}(x)\| \le \frac{1-\mu}{128} \|h_{ki}(x) - h_{k\,i-1}(x)\|$. Therefore $\| [R_{kx}(h_{ki}(x)) - R_{kx}(h_{k\,i-1}(x))] + [f_k(x + h_{k\,i}(x) - f_k(x + h_{k\,i-1}(x))] \| \le \|R_{kx}(h_{ki}(x)) - R_{kx}(h_{k\,i-1}(x))\| + \| f_k(x + h_{k\,i}(x) - f_k(x + h_{k\,i-1}(x)) \| \le \frac{1-\mu}{8} \|h_{ki}(x) - h_{k\,i-1}(x)\| + \frac{1-\mu}{128} \|h_{ki}(x) - h_{k\,i-1}(x)\| < \frac{1-\mu}{4} \|h_{ki}(x) - h_{k\,i-1}(x)\|$ for a component k of the $F_i - F_{i-1}$ at i = 1, 2, …, m . In other words, $\| F_i - F_{i-1} \|_B < \frac{1-\mu}{4} \cdot \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B$. Therefore $\| \mathbf{h}_{i+1} - \mathbf{h}_i \|_B = \|L^{-1}(F_i) - L^{-1}(F_{i-1})\|_B = \|L^{-1}(F_i - F_{i-1})\|_B \le \|L^{-1}\|_B \cdot \|F_i - F_{i-1}\|_B < \frac{2}{1-\mu} \cdot \frac{1-\mu}{4} \cdot \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B = \frac{1}{2} \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B$. It means that the iterative process is contracting with a compression constant $< \frac{1}{2}$ for i = 1, 2, …, m.

Thus, $\| \mathbf{h}_1 - \mathbf{h}_0 \|_B > \frac{1}{2} \| \mathbf{h}_2 - \mathbf{h}_1 \|_B > \ldots > \frac{1}{2^m} \| \mathbf{h}_{m+1} - \mathbf{h}_m \|_B$. Therefore $\| \mathbf{h}_{m+1} \|_B = \| \mathbf{h}_{m+1} - \mathbf{h}_0 \|_B \le \| \mathbf{h}_{m+1} - \mathbf{h}_m \|_B + \| \mathbf{h}_m - \mathbf{h}_{m-1} \|_B + \ldots + \| \mathbf{h}_1 - \mathbf{h}_0 \|_B < (1 + \frac{1}{2} + \ldots + \frac{1}{2^m}) \| \mathbf{h}_1 - \mathbf{h}_0 \|_B < 2 \| \mathbf{h}_1 \|_B < 2 \frac{\alpha}{2} = \alpha$, that it is required. So, all $\mathbf{h}_i$ do not leave the neighborhood O and $\| \mathbf{h}_{i+1} - \mathbf{h}_i \|_B \le \frac{1}{2} \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B$ (i = 1, 2, …) . Hence, according to a principle of contracting maps, there exists lim $\mathbf{h}_i = \mathbf{h} \in \mathbf{B}$, which is a solution of (9).

Continuous maps $H_k(x) = x + h_k(x)$ are homeomorphisms if they are injective. But $H_k(x_1) = x_1 + h_k(x_1) = H_k(x_2) = x_2 + h_k(x_2) \Rightarrow x_2 - x_1 = h_k(x_1) - h_k(x_2) \Rightarrow \|x_2 - x_1\| < 2\alpha$, as $\| \mathbf{h}_k \|_B < \alpha$. From equality $\tilde{G}_k \circ H_k = H_{k+1} \circ G_k$ follows that $\tilde{G}_{k+m} \circ \ldots \circ \tilde{G}_{k+1} \circ \tilde{G}_k \circ H_k = H_{k+m} \circ G_{k+m} \circ \ldots \circ G_k$ by the commutative diagram. If $x_2 - x_1$ has a nonzero component in stretching foliation at large enough m the distance between $G_{k+m} \circ \ldots \circ G_k(x_2)$ and $G_{k+m} \circ \ldots \circ G_k(x_1)$ will exceed 2α, so $H_{k+m} \circ G_{k+m} \circ \ldots \circ G_k(x_2) \ne H_{k+m} \circ G_{k+m} \circ \ldots \circ G_k(x_1)$. But $H_{k+m} \circ G_{k+m} \circ \ldots \circ G_k(x_2) = \tilde{G}_{k+m} \circ \ldots \circ \tilde{G}_{k+1} \circ \tilde{G}_k \circ H_k(x_2) = \tilde{G}_{k+m} \circ \ldots \circ \tilde{G}_{k+1} \circ \tilde{G}_k \circ H_k(x_1) = H_{k+m} \circ G_{k+m} \circ \ldots \circ G_k(x_1)$. If $x_2 - x_1$ belongs contracting foliation the similar reasoning leads to a contradiction to the supposition, that $H_m(x_1) = H_m(x_2)$. The theorem is proved.

The important feature of this variant of the theorem about sets of ε – trajectories is its constructability, presence of explicit estimations, which can be used at the numerical analysis of concrete systems. These formulas essentially become simpler in the most important for applications case, when the compact set K from a theorem condition is DQM attractor and by that is the closure of some invariant area of DQM in M.

In this case let O is a neighborhood of DQM attractor K, μ is a coefficient of hyperbolicity on O: $\|DG_{kx}(h_{ks}(x)\| < \mu \|h_{ks}(x)\|$ and $\|DG_{kx}(h_{ku}(x)\| > \frac{1}{\mu}\|h_{ku}(x)\|$ for an any continuous vector – function $h_k(x)$ on O, where $h_k(x) = h_{ks}(x) + h_{ku}(x)$ is a decomposition of $h_k(x)$ on stratifications. Then, as shown in the proof of theorem 5, operator L on Banach space **B** of the sequences of regularly limited continuous vector – functions $h_k(x)$ on O is reversible and $\|L^{-1}\|_B < \frac{1}{1-\mu}$. Consequently a solution of the system of homologous equations (10) is $\mathbf{h}_1 = L^{-1}(f)$ and $\| \mathbf{h}_1 \|_B \le \| L^{-1}(\boldsymbol{f}) \|_B \le \| L^{-1} \|_B \cdot \| \boldsymbol{f} \|_B \le \frac{1}{1-\mu} \cdot \| \boldsymbol{f} \|_B$. Here $\mathbf{h}_1$ is first iteration of the contracting maps given by the formula $\mathbf{h}_i = L^{-1}(F(\mathbf{h}_{i-1}))$, where $F(\mathbf{h}) = (\ldots, R_{-1x}(h_{-1}(x)) + f_{-1}(x + h_{-1}(x)), R_{0x}(h_0(x)) + f_0(x + h_0(x)), R_{1x}(h_1(x)) + f_1(x + h_1(x)), \ldots) \in \mathbf{B}$. The convergence proof is spent on an induction: from $\|\mathbf{h}_i\|_B \le a$ at i = 1, 2, …, m, where $a = 2\|\mathbf{h}_1\|_B$ we obtain, that $\|\mathbf{h}_{m+1}\|_B \le a$ and $\| \mathbf{h}_{i+1} - \mathbf{h}_i \|_B \le \frac{1}{2} \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B$ $(1 \le i \le m)$. Here $\| \mathbf{h}_{i+1} - \mathbf{h}_i \|_B \le \|L^{-1}(F_i) - L^{-1}(F_{i-1})\|_B \le \|L^{-1}\|_B \cdot \|F_i - F_{i-1}\|_B \le \frac{1}{1-\mu} \cdot \|F_i - F_{i-1}\|_B$, and component k of vector $F_i - F_{i-1} \in \mathbf{B}$ is equal to

$$[R_{kx}(h_{k\,i}(x)) - R_{kx}(h_{k\,i-1}(x))] + [f_k(x + h_{k\,i}(x) - f_k(x + h_{k\,i-1}(x))].$$

As

$$\|R_{kx}(h_{ki}(x)) - R_{kx}(h_{k\,i-1}(x))\| \le 4\|G(x)\|_2 \cdot \max\{\|h_{ki}(x)\|, \|h_{k\,i-1}(x)\|\}\|h_{ki}(x) - h_{k\,i-1}(x)\|$$

by (12) and $\| f_k(x + h_{k\,i}(x) - f_k(x + h_{k\,i-1}(x)) \| \le \|f_k\|_1 \cdot \|h_{ki}(x) - h_{k\,i-1}(x)\|$ then

$$\| \mathbf{h}_{i+1} - \mathbf{h}_i \|_B \le \|L^{-1}\|_B \cdot \|F_i - F_{i-1}\|_B \le \frac{1}{1-\mu} \cdot (4\|G(x)\|_2 \cdot a + \| \boldsymbol{f} \|_B) \cdot \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B$$

$$\le \frac{1}{1-\mu} \cdot (8\|G(x)\|_2 \cdot \frac{1}{1-\mu} \cdot \| \boldsymbol{f} \|_B + \| \boldsymbol{f} \|_B) \cdot \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B$$

$$\| \mathbf{h}_{i+1} - \mathbf{h}_i \|_B \le \frac{1}{1-\mu} \| \boldsymbol{f} \|_B (\frac{1}{1-\mu} 8\|G(x)\|_2 + 1) \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B$$

for i = 1, 2, …, m. Then $\| \mathbf{h}_{i+1} - \mathbf{h}_i \|_B \le \frac{1}{2} \| \mathbf{h}_i - \mathbf{h}_{i-1} \|_B$, if $\| \boldsymbol{f} \|_B \le \frac{(1-\mu)^2}{4(4\|G\|_2 + 1)}$.

In this case, in view of $\mathbf{h}_0 = 0$, $\| \mathbf{h}_{m+1} \|_B = \| \mathbf{h}_{m+1} - \mathbf{h}_0 \|_B \le \| \mathbf{h}_{m+1} - \mathbf{h}_m \|_B + \ldots + \| \mathbf{h}_1 - \mathbf{h}_0 \|_B < (1 + \ldots + \frac{1}{2^m}) \| \mathbf{h}_1 - \mathbf{h}_0 \|_B < 2 \| \mathbf{h}_1 \|_B = a$, as it is required. From this

Corollary 1. Let $\tilde{G}_k(x)$ be a hyperbolic ε – realization of DQM realization $G_k(x)$ on DQM attractor K: $\| \tilde{G}_k(x) - G_k(x) \|_{C^1} \le \varepsilon$. Then $\tilde{G}_k(x)$ is topologically equivalent to $G_k(x)$, i.e. there are the homeomorphisms H_k such that: $\tilde{G}_k \circ H_k = H_{k+1} \circ G_k$ (k = 0, ±1, ±2, …),

$$\varepsilon \le \frac{(1-\mu)^2}{4(4\|G\|_2 + 1)} . \tag{13}$$

The program for finding and analysis of DQM attractors.

In this work the program is used, which 1) builds DQM discretization s and discovers their attractors; 2) checks the conditions, used at the proof of hyperbolicity of dynamics on a DQM attractor. This program is realized as C # – application with use of Open Maple technology and consists of 4 basic procedures (modules).

1. Prestep. Procedure to each cell Δ_i from a define area $\Omega = \{\Delta_i\}$ of DQM discretization puts in correspondence the cells – images for one step of dynamics. In other words, this procedure sets on Ω a topological Markov chain.

2. Findattr. Procedure finds in Ω coherent components of attractor of the topological Markov chain set in Prestep.

3. Hyperproc. Procedure checks on discovered in Findattr components the assumptions used at the proof of hyperbolicity of dynamics on them.

4. Animate. Procedure allows to localize area Ω for Prestep procedure and visualizes behavior of system, using animation technology in Maple.

Let's describe the basic algorithm of this program used in procedure Findattr. Further the detailed flowchart of this procedure is cited (Figure 1).

Let's the partition of area Ω on cells Δ_i is already set by procedure Prestep. Let's Prestep procedure sets symbolical dynamics of the topological Markov chain H on Ω, which status are cells Δ_i. We will consider a quasiorder transitive relation: $\Delta_i \prec \Delta_j$ on the space of status $\Omega = \{\Delta_i\}$, if there is some trajectory of symbolical dynamics from Δ_i in Δ_j. The status Δ_i is reflexive, if $\Delta_i \prec \Delta_i$. Reflexive status are divided into equivalence classes:

$$\Delta_i \sim \Delta_j \Leftrightarrow \Delta_i \prec \Delta_j \prec \Delta_i .$$

Then $H(\Omega) \supset H^2(\Omega) \supset H^3(\Omega) \supset \ldots \supset H^n(\Omega)$ for area $\Omega = \{\Delta_i\}$. If $H^n(\Omega) = H^{n+1}(\Omega)$ then $H^n(\Omega)$ is an attractor of DQM discretization. The following algorithm is based on it.

Procedure Findattr contains 5 parameters (arguments):

1) n is a number of the first considered cell.

2, 3) nx, ny are length and width of the rectangle Ω, where Ω is the procedure define area, expressed in quantities of cells.

4) del. If the cell Δ_i is contained at an image of a cell Δ_j then at this image there are also cells at the right, at the left, below and above from a cell Δ_i with the quantity del (by DQM definition).

5) M is a matrix received from procedure Prestep. Its each string i contains the data about an image of a cell Δ_i, it has eight columns. The first column in M contains the number of cell Δ_i in the matrix R, which is a resultant for Findattr. The eighth column contains a number of first still not considered cell from an image of a cell Δ_i. The others 6 columns contain coordinates of cells in image of a cell Δ_i, to which three tops of a cell Δ_i get: lower left and two adjacent with it.

Results of Findattr are written in matrix R containing 2 columns. The first column contains number of an equivalence class of reflexive states for the given cell. The second one contains a number of this cell in the initial matrix M.

Further we will use the detailed flowchart of Findattr procedure (Figure 1).

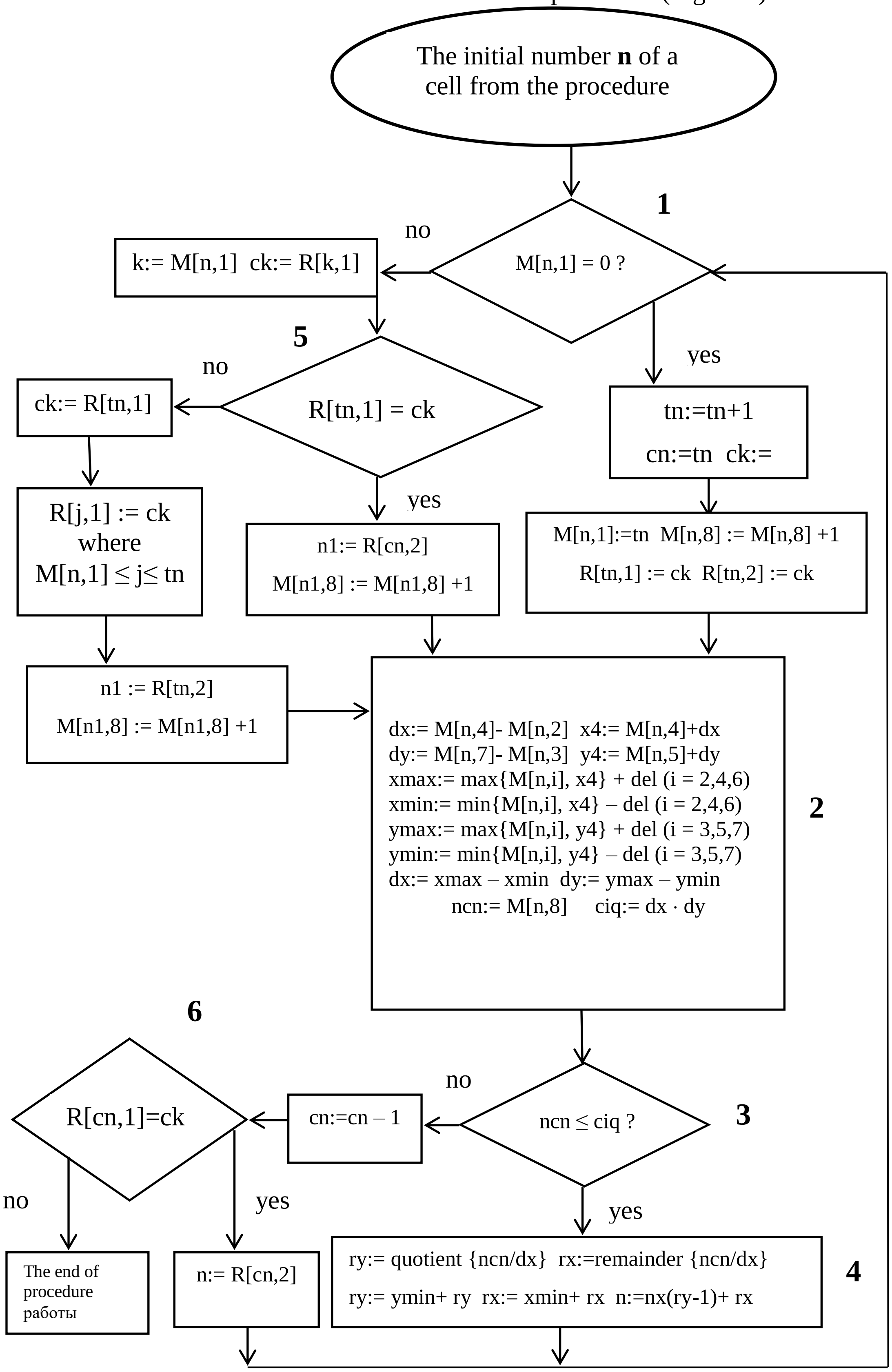

Figure 1.

Here the variable *cn* sets a cell current number, the variable *tn* is a total number of already considered cells, the variable *ck* is a number of an equivalence class of reflexive states for a current cell.

In the block flagged in digit **1**, for each cell it is checked, whether it is written already in matrix R (in other words, whether it was handled already by Findattr procedure). If still is not, then class number *ck* increases on 1 and this value is written in matrix R for this cell. Then in the block flagged in digit **2** the set of images of the given cell *cn* is restored. There x4, y4 are coordinates of a cell, that get an image of the upper right top of a cell *cn*; *xmax*, *xmin*, *ymax*, *ymin* are the coordinates of tops of the rectangle containing all images of the cell *cn*; *dx*, *dy* are the length and the width of this rectangle. Then *ncn* is a number of first still not considered cell from the image of the cell *cn*; $ciq = dx \cdot dy$ is a total number of cells in a rectangle of images.

In the block flagged in digit **3** it is checked, whether there exists some cells from the image of a current cell *cn*, that still are not considered by Findattr procedure. If yes, then in the block flagged in digit **4** we find coordinates *rx, ry* of the first such cell; then its number n; and then we come back to the block **1**.

If while checking the block **1** it appears that the current cell *cn* already was considered by Findattr procedure, then in the block **5** we check, whether it has current, i.e. the greatest number of a class *ck*. If yes, then for this cell we select number of first still not considered cell from its image; and we come back with it in the block **2**. But if it has appeared in block **5,** that class number is less, than the current one *ck*, then besides we assign this greatest class number *ck* for all cells between this and last one in matrix R. Really, it means that all these cells are in one equivalence class of reflexive states.

At last, if in the block **3** it is clarified that all images of the given cell *cn* are already considered, then in matrix R we pass to the previous cell. If in the block **6** it appears that it has current (that is the greatest) number of a class, then with number of this cell we come back to the beginning in the block **1**. But if it has appeared that class number for the previous cell is less than the greatest one, then the procedure is completed. Really, on procedure constructions, for all previous cells all the images long are already considered. But then it means that all previous cells from R belong to maximum class of an equivalence of reflexive states; and this cell does not concern it any more.

Example of DQM method: study of Henon system.

We will consider here DQM method for investigation of dynamics of concrete systems on an example of two-dimensional system of M. Henon [6]: $(x, y) \to (1 + y - ax^2, bx)$. Values of parameters $a = 1.7, b = 0.5$ we will choose those, at which for system of R. Lozi [11]: $(x, y) \to (1 + y - a|x|, bx)$ presence of an attractor with hyperbolic dynamics has been strictly proved.

1. Animate Procedure localizes area Ω of a phase space, in which the system attractor hypothetically contains. On the basis of outcomes of the numerical researches, visually presented further in a Figure 2, we choose a rectangle $\Omega = \{(x, y) \mid -1 \le x \le 1.5; -0.1 \le y \le 0.1\}$.

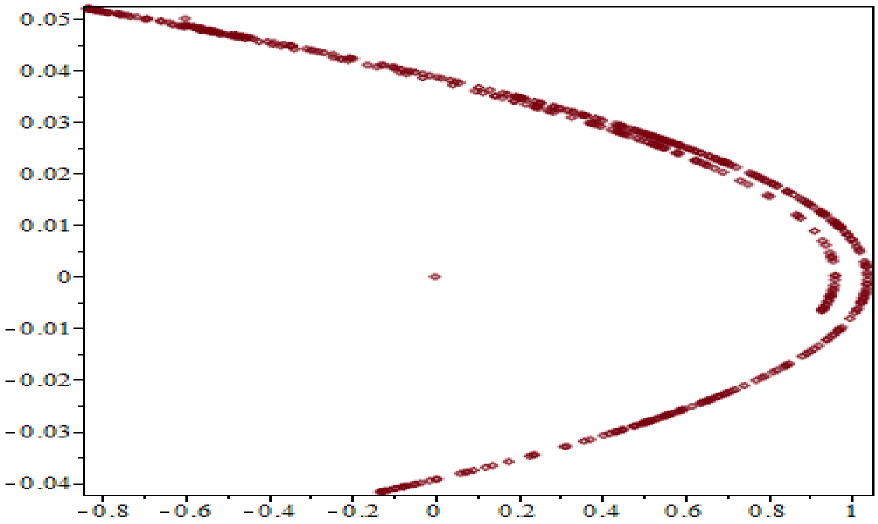

Figure 2.

Outcomes of work of Animate procedure lead also to the supposition that on an attractor in Ω the system is hyperbolical: in Figure 3 for iteration $n = 1, 2, \ldots, 500$ its coordinate $x \in \Omega$ answers.

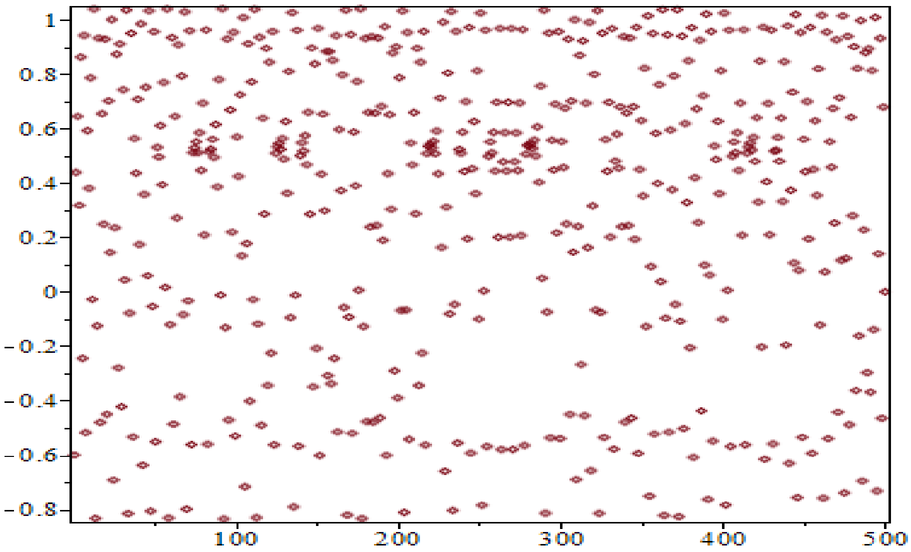

Figure 3.

In Figure 4 for iteration $n = 1, 2, \ldots, 500$ its coordinate $y \in \Omega$ answers.

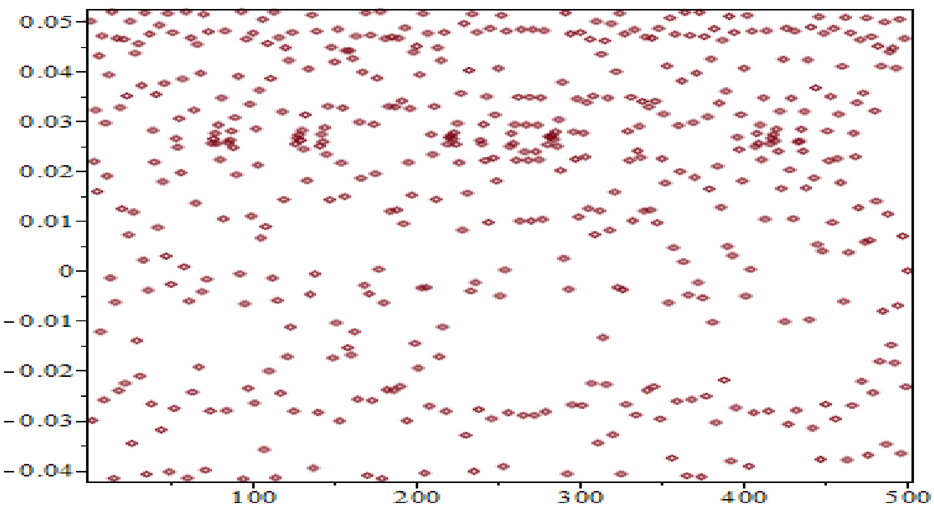

Figure 4.

2. Prestep Procedure makes rectangle splitting Ω on cells Δ_i, i.e. squares with the sides of length 0.01, parallel to coordinate axes. Then Δ_i Prestep puts each cell in correspondence to the collection of cells to which points from Δ_i can get for one step of dynamics of Henon system. It is simultaneously strictly established, that the area Ω is really invariant with respect to DQM discretization given by the splitting of Ω. Thereby this procedure sets the topological Markov chain which space of status is the set of cells $\Delta_i \subset \Omega$.

3. Findattr Procedure finds in Ω a DQM attractor for given in Prestep DQM discretization according to described above its algorithm. The attractor appears coherent that corresponds to the data obtained by Animate procedure (Figure 2).

4. The basic outcomes of investigation are connected with following corollary from the theorem 5 (and from corollary 1 of it), which statement is oriented especially on use at study of concrete dynamic systems.

Corollary 2. Let's Δ_i be the cells of attractor of DQM ε – discretization of the system, given by a two-dimensional diffeomorphism G, $x_i \in \Delta_i$ $(1 \le i \le N)$. Let's the average eigenvalues $\lambda_1(x_i)$ and $\lambda_2(x_i)$ of differential DG for m iterations G from a point x_i (i.e. of differentials in points x_i, $G(x_i)$, $G^2(x_i)$, …, $G^{m-1}(x_i)$) satisfy the conditions $\lambda_1(x_i) < \mu$, $\lambda_2(x_i) > \frac{1}{\mu}$ for some μ $(0 < \mu < 1)$ in any point $x_i \in \Delta_i$ $(1 \le i \le N)$ and

$$\varepsilon \le \frac{(1-\mu)^2}{4m(4\|G\|_2+1)} . \tag{14}$$

Then

1) the initial system given by diffeomorphism G, is hyperbolical on the attractor;

2) any DQM ε – realization of this system is also hyperbolical on DQM attractor and is topologically equivalent to initial system;

3) the support of the attractor of initial system and its DQM attractor coincide with accuracy of order ε.

Here by an attractor of initial system we mean intersection $O \cap G(O) \cap G^2(O) \cap \ldots \cap G^n(O) \cap \ldots$ for some neighborhood O of an attractor of the DQM ε – discretization from collorary 2. The value

$$\|G\|_2 = \max_{\Omega}\{1, \sqrt{(\frac{\partial G}{\partial x})^2 + (\frac{\partial G}{\partial y})^2}, \sqrt{(\frac{\partial^2 G}{\partial x^2})^2 + 2(\frac{\partial^2 G}{\partial x \partial y})^2 + (\frac{\partial^2 G}{\partial y^2})}\} ,$$

where for diffeomorphism $G(x, y) = (X(x, y), Y(x, y))$ we set $(\frac{\partial G}{\partial x})^2 = (\frac{\partial X}{\partial x})^2 + (\frac{\partial Y}{\partial x})^2$, $(\frac{\partial^2 G}{\partial x^2})^2 = (\frac{\partial^2 X}{\partial x^2})^2 + (\frac{\partial^2 Y}{\partial x^2})^2$ and so on, and the domain Ω of a phase space contains a system attractor. In a corollary 2 we were limited to a two-dimensional case though its multidimensional generalization is also true.

5. In our case, when G(x, y) sets the system of Henon, on a rectangle $\Omega = \{(x, y) \mid -1 \le x \le 1.5;\ -0.1 \le y \le 0.1\}$ is fulfilled

$$\|G\|_2 = \max_{\Omega}\{1, \sqrt{(2ax)^2 + b^2} + 1, 2a\} \approx 6{,}1.$$

Hyperproc Procedure establishes, that for m = 10 average eigenvalues $\lambda_1(x_i)$ and $\lambda_2(x_i)$ of differential DG for m iterations G satisfy conditions $\lambda_1(x_i) < 0.4$, $\lambda_2(x_i) > 1.7$ for all x_i. Value $1/1.7 \approx 0.59$. Thus, $\mu \ge 0.59$; however we choose value $\mu = 0.7$ with a large supply: the reason will clear up further. Then under the formula (14) $\varepsilon \approx 0.0001$.

6. Now it is necessary to repeat sequence of operations since item 2 with that only difference, that rectangle splitting Ω on cells Δ_i contains squares with the sides of length not 0.01, but 0.0001. The main size of calculations is necessary just on this stage of research. Therefore all procedures of the program complex assume the possibility of definition of their operating time

and saving the subproducts received in this time. It is possible further to continue work with saving subproducts or to adjust the selected options, as a result of the analysis of these subproducts.

Eventually, the purpose of all these evaluations is to check up that for DQM ε – discretization with this new smaller ε inequalities $\lambda_1(x_i) < \mu$ and $\lambda_2(x_i) > \frac{1}{\mu}$ are still fulfilled with the same μ for all cells Δ_i from an attractor of ε – discretization . In this case for this ε – discretization all assumptions of a corollary are automatically satisfied 2. Otherwise all this cycle of calculations since item 2 is necessary to repeat, preliminary having specified parameters. On purpose to avoid it we have been selected the value of parameter μ = 0.7 with a store in item 5. In our case at the chosen parameters check has passed successfully.

Thereby in this case all conclusions of a corollary 2 are true. We already obtained the structure of the topological Markov chain of DQM ε – discretization, as a result of evaluations. In view of the theorem 2, it gives us the detailed and strictly proved information on dynamics of Henon system on its attractor.

The chosen values of parameters a = 1.7 and b = 0.5 are not unique. For example, similar outcomes turn out at a = 1.4 and b = 0.35. In following figures a view of attractor (Figure 5) and dynamics visualization on this attractor on axis OX (Figure 6) and axis OY (Figure 7) are shown.

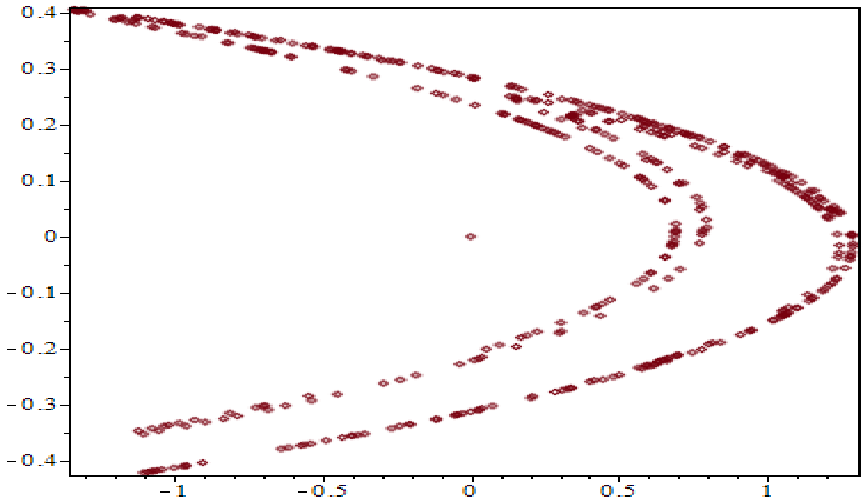

Figure 5.

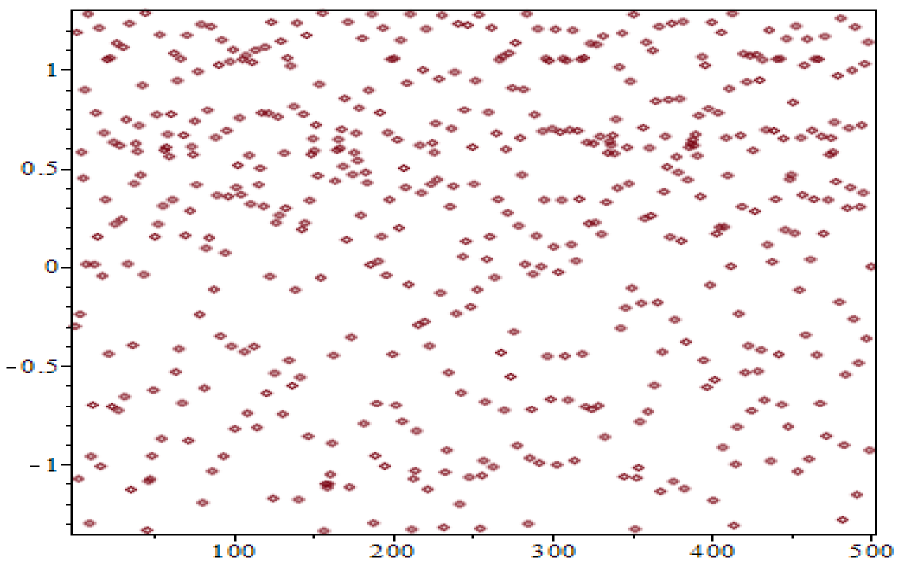

Figure 6.

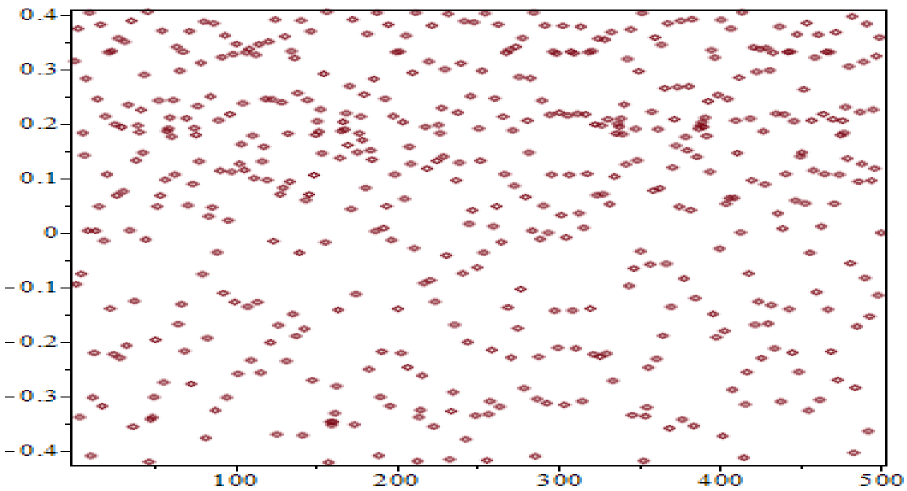

Figure 7.

Conclusions. The DQM method for investigating the dynamics of concrete systems and obtaining strict results is demonstrated on the example of M. Henon's system. We choose the values of parameters at which this system is hyperbolical on the attractor; we determine the support of this "strange attractor" within given error and the dynamics on it within topological equivalence. The program realized as the C# – application with usage of Open Maple technology is used here. In this work the purpose is only the illustration of DQM method for study of concrete dynamic systems. The detailed statement of results is supposed in the subsequent publications.

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Стаття надійшла до редакції

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ЧИСЕЛЬНИЙ АНАЛІЗ ДИНАМІЧНОЇ СИСТЕМИ ТА ЇЇ СТРУКТУРНА СТІЙКІСТЬ

У статті продемонстровано метод дослідження динаміки конкретних систем малої вимірності та отримання математично точних результатів на прикладі системи М. Хеннона. Відповідна програма реалізована, як С# – додаток із застосуванням технології Open Maple.

Вона дозволяє знаходити атрактори динамічних систем малої вимірності та доводити гіперболічну поведінку на них, використовуючи обчислення на комп'ютері. Проте, таким чином отримуємо точні апостеріорні результати, що ґрунтуються на теоремах цієї статті. Комп'ютерні обчислення використовуються для перевірки виконання умов цих тверджень.

Можливість отримання математично обґрунтованих результатів чисельних досліджень пов'язана з структурною стійкістю застосованої моделі. Структурна стійкість є базовою концепцією двох традиційних університетських курсів: “Математичне моделювання та системний аналіз” і “Методи обчислень”. Автором запропонований підхід, що дозволяє для кожної даної динамічної системи побудувати стійку модель. Для цього виявляється достатнім розглядати цю систему разом з випадковими флуктуаціями, неусувними для кожної реальної системи. Точніше кажучи, для даної класичної системи будуємо її збурення певним марковським процесом, який називаємо динамічною квантовою моделлю (ДКМ) цієї системи. Така модель є стійкою, що забезпечує можливість її чисельного дослідження. А з наближенням флуктуацій до нуля динаміка ДКМ прямує до динаміки заданої класичної системи.

Ключові слова. динамічний, система, квантова, структурний, теорія, алгоритм, атрактор.

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ЧИСЛЕННЫЙ АНАЛИЗ ДИНАМИЧЕСКОЙ СИСТЕМЫ И ЕЁ СТРУКТУРНАЯ УСТОЙЧИВОСТЬ

В статье продемонстрирован метод исследования динамики конкретных систем малой размерности и получения математически строгих результатов на примере системы М. Хеннона. Соответствующая программа реализована, как C# приложение с использованием технологии Open Maple. Она позволяет находить аттрактор динамической системы малой размерности и доказывать его гиперболичность, используя вычисления на компьютере. Однако, таким образом получены точные апостериорные результаты, основанные на теоремах этой статьи. Компьютерные вычисления использованы для проверки условий этих утверждений.

Возможность получения математически обоснованных результатов численных исследований связана со структурной устойчивостью используемой модели. Структурная устойчивость является базовым понятием двух традиционных университетских курсов: “Математическое моделирование и системный анализ” и “Методы вычислений”. Автором предложен подход, который позволяет для каждой заданной динамической системы построить устойчивую модель. Для этого достаточным оказывается рассматривать эту систему вместе со случайными флуктуациями, неустранимыми для любой реальной системы. Точнее говоря, для данной классической системы строим её возмущение марковским процессом, называемым динамической квантовой моделью (ДКМ) этой системы. Такая модель устойчива, что обеспечивает возможность её численного исследования. А при стремлении флуктуаций к нулю динамика ДКМ сходится к динамике данной классической системы.

Ключевые слова. Динамический, система, квантовая, структурный, теория, алгоритм, аттрактор.