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Eigenvalue Distribution of Bipartite Large Weighted Random Graphs. Resolvent Approach

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We study an eigenvalue distribution of the adjacency matrix $A^{(N,p,\alpha)}$ of the weighted random bipartite graphs $\Gamma = \Gamma_{N,p,\alpha}$. We assume that the graphs have N vertices, the ratio of parts is $\dfrac{\alpha}{1-\alpha}$, and the average number of edges attached to one vertex is αp for the first part and $(1-\alpha)p$ for the second part of vertices. To each edge of the graph e_{ij}, we assign the weight given by a random variable a_{ij} with the finite second moment.

We consider the resolvents $G^{(N,p,\alpha)}(z)$ of $A^{(N,p,\alpha)}$ and study the functions

$$f_{1,N}(u,z) = \frac{1}{[\alpha N]} \sum_{k=1}^{[\alpha N]} e^{-ua_k^2 G_{kk}^{(N,p,\alpha)}(z)}$$

and

$$f_{2,N}(u,z) = \frac{1}{N-[\alpha N]} \sum_{k=[\alpha N]+1}^{N} e^{-ua_k^2 G_{kk}^{(N,p,\alpha)}(z)}$$

in the limit $N \to \infty$. We derive a closed system of equations that uniquely determine the limiting functions $f_1(u,z)$ and $f_2(u,z)$. This system of equations allows us to prove the existence of the limiting measure $\sigma_{p,\alpha}$. The weak convergence in probability of the normalized eigenvalue counting measures is proved.

Key words: sparse random matrices, bipartite graphs, normalized eigenvalue counting measure.

Mathematics Subject Classification 2010: 15B52.

1. Introduction

Random graphs appear in different branches of mathematics and physics (see monographs [5, 7] and references therein). It is well known that they are closely connected with the theory of random matrices, since there is a one to one map between the graphs with N vertices and their adjacency matrices (recall that by definition the entries a_{ij} of the adjacency matrix are 1 if the vertices i and j are connected, and $a_{ij} = 0$ otherwise). The spectrum of the graph is the set of n eigenvalues of the adjacency matrix.

The model of a random graph, introduced by P. Erdos (see, e.g., [7] and references therein), is a simple but very important and popular sample of random graphs. In this model, usually called the $G(N, p_N)$ random graph, each pair of N graph vertices is connected by the edge with probability p_N, independently from all other edges of the graph. The corresponding ensemble of random $N \times N$ adjacency matrices M can be represented as $M = \{m_{ij}\}_{i,j=1}^N$ with $m_{ii} = 0$ and i.i.d.

$$m_{ij} = \begin{cases} 1, & \text{with probability } \ p_N, \\ 0, & \text{with probability } 1 - p_N. \end{cases} \tag{1.1}$$

The ensemble of adjacency matrices (1.1) is a particular case of the random matrix ensemble. The eigenvalue distribution of different ensembles of $N \times N$ random matrices, as $N \to \infty$, was intensively studied during a half of century (see [10] and references therein) since the pioneer work of Wigner [17]. One of the questions of primary interest is the asymptotic behavior, as $N \to \infty$, of the so-called Normalized Eigenvalue Counting Measure (NCEM) defined by the formula

$$N_n(\lambda) = \frac{\sharp\{j : \lambda_j^{(N)} < \lambda\}}{N},$$

where $\{\lambda_j^{(N)}\}_{j=1}^N$ are the eigenvalues of the matrix under consideration. Ensemble (1.1) plays an important role in the random matrix theory not only because of its links with random graphs, but also because, varying p_N from $O(1/N)$ to $O(1)$ (as $N \to \infty$), one can interpolate between the matrices of Wigner type, which have almost all entries nonzero, and the so-called sparse random matrices with $p_N = O(1/N)$. The intermediate case, when $1 << p_N << 1$, is also interesting, and it is studied both in physics and mathematics. In particular, in [9], the limiting eigenvalue distribution of ensemble (1.1) was found, and it appears that the distribution is the same as that for the classical Wigner model [17]. The most interesting sparse regime ($p_N = O(1/N)$) for ensembles (1.1) was studied intensively first in physics literature, where the convergence of NCEM to some non random limiting measure was established on the physical level of rigor (see [12, 13]). Then in the papers [2, 3] the same results were obtained on the mathematical level of rigor by using the so-called moment approach based on the convergence of the NCEM moments to some non random limits. In [8], the similar results for NCEM were obtained for the case of the so-called weighted random graphs, which differ from (1.1) by some random independent from $\{m_{ij}\}_{1\le i<j\le N}$ multipliers $\{a_{ij}\}_{1\le i<j\le N}$, usually called weights. The spectrum for sparse random graphs with subgaussian weigths coincides with the whole real line. Also, in this case the tails decay as $\exp\{-cx^2 log(x)\}$ as $x\to\infty$. The ensembles of weighted random graphs are now especially interesting because of their links with the so-called ensembles with blowing moments and also ensembles of matrices with heavy tales [4]. The next step in studying the eigenvalue distribution of large random graphs (the proof of Central Limit Theorem for linear eigenvalue statistics) was done in [14] for the sparse regime and in [15] for the diluted regime.

In the present paper, we consider the bipartite analogs of large sparse random graphs. The moment approach for this ensemble was developed in [16], where it was shown that the moments of the corresponding NCEM have the limits which satisfy the system of recursive relations. But this approach requires that rather restrictive conditions be imposed on the moments of the weights $\{a_{ij}\}_{1\le i<j\le N}$. We apply the resolvent approach, developed in [8] for the case of standard random weighted graphs, to the case of bipartite graphs. This allows us to optimize the conditions imposed on the moments of $\{a_{ij}\}_{1\le i<j\le N}$.

2. Main Results

Let us introduce the randomly weighted adjacency matrix of random bipartite graphs. Let $\Xi=\{a_{ij},\ i\le j,\ i,j\in\mathbb{N}\}$ be the set of jointly independent identically distributed (i.i.d.) random variables determined on the same probability space. We set $a_{ji}=a_{ij}$ for $i\le j$.

Given $0<p\le N$, let us define the family $D_N^{(p)}=\{d_{ij}^{(N,p)},\ i\le j,\ i,j\in\overline{1,N}\}$ of jointly independent random variables

$$d_{ij}^{(N,p)}=\begin{cases}1, & \text{with probability } \ p/N,\\ 0, & \text{with probability } 1-p/N.\end{cases}\tag{2.1}$$

We set $d_{ji}=d_{ij}$ and assume that $D_N^{(p)}$ is independent from Ξ.

Let $\alpha\in(0,1)$, define $I_{\alpha,N}=\overline{1,[\alpha\cdot N]}$, where $[\cdot]$ is an integer part of the number. Now one can consider the real symmetric $N\times N$ matrix $A^{(N,p,\alpha)}(\omega)$:

$$\left[A^{(N,p,\alpha)}\right]_{ij}=\begin{cases}a_{ij}d_{ij}^{(N,p)}, & \text{if } (i\in I_{\alpha,N}\wedge j\notin I_{\alpha,N})\vee(i\notin I_{\alpha,N}\wedge j\in I_{\alpha,N}),\\ 0, & \text{otherwise}\end{cases}\tag{2.2}$$

that has N real eigenvalues $\lambda_1^{(N,p,\alpha)}\le\lambda_2^{(N,p,\alpha)}\le\ \ldots\le\ \lambda_N^{(N,p,\alpha)}$.

The normalized eigenvalue counting function of $A^{(N,p,\alpha)}$ is determined by the formula

$$\sigma\left(\lambda; A^{(N,p,\alpha)}\right) = \frac{\sharp\{j : \lambda_j^{(N,p,\alpha)} < \lambda\}}{N}.$$

We denote by $\mathcal{F}$ the class of functions which are analytic with respect to z : $\Re z > 0$ and for any fixed z : $\Re z > 0$ possessing the norm

$$||f(u,z)|| = \sup_{u>0} \frac{|f(u,z)|}{\sqrt{1+u}}. \tag{2.3}$$

Theorem 1. *Assume that the $\mu(a) = \mathbb{E}\{\theta(a - a_{i,j})\}$ probability distribution of $a_{i,j}$ possesses the property*

$$\int a^4 d\mu(a) = X_4 < \infty. \tag{2.4}$$

(We denote by X_i the i-th absolute moment of a, i.e., $X_i = \int |a|^i d\mu(a)$.) Then the measure $\sigma\left(\lambda; A^{(N,p,\alpha)}\right)$ converges weakly in probability to nonrandom measure $\sigma_{p,\alpha}$,

$$\sigma\left(\cdot\, ; A^{(N,p,\alpha)}\right) \stackrel{w,P}{\to} \sigma_{p,\alpha}, \;\; N \to \infty. \tag{2.5}$$

The Stieltjes transform $g_{\sigma_{p,\alpha}}$ of the limiting measure $\sigma_{p,\alpha}$ can be found as follows:

$$g_{\sigma_{p,\alpha}}(z) = -ih(iz), \tag{2.6}$$

$$h(z) : \mathbb{C}_+ \to \mathbb{C}_+, \;\; h(z) = \alpha \cdot h_1(z) + (1-\alpha) \cdot h_2(z), \tag{2.7}$$

$$h_i(z) = -X_2^{-1} \frac{\partial}{\partial u} f_i(u,z)\Big|_{u=0}, \;\; i = 1,2, \tag{2.8}$$

where a pair $f_1(u,z)$ and $f_2(u,z)$ is a unique solution of the following system of functional equations in the class $\mathcal{F}$:

$$\begin{cases} f_1(u,z) = L(f_2, \mu, 1-\alpha) \\ f_2(u,z) = L(f_1, \mu, \alpha) \end{cases}, \tag{2.9}$$

where

$$L(f,\mu,\alpha) = 1 - u^{1/2} e^{-\alpha p} \int |a| d\mu(a) \int_0^\infty dv \frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}} \exp\{-zv + \alpha p f(v,z)\}, \tag{2.10}$$

$\mathcal{J}_1(\zeta)$ is the Bessel function

$$\mathcal{J}_1(\zeta) = \frac{\zeta}{2} \sum_{k=0}^{\infty} \frac{(-\zeta^2/4)^k}{k!(k+1)!}. \tag{2.11}$$

Proposition 1. *Condition* (2.4) *in Theorem* 1 *can be replaced by*

$$\int a^2 d\mu(a) = X_2 < \infty$$

via the truncation procedure.

The proof of Proposition 1 is given in Section 4..
Theorem 1 is a corollary of Theorem 2.

Theorem 2. *Let the distribution of $a_{j,k}$ satisfy condition* (2.4). *Then* (*i*) *the variance of $g_{N,p,\alpha}(z)$ vanishes in the limit $N \to \infty$:*

$$\mathbb{E}\{|g_{N,p,\alpha}(z) - \mathbb{E}\{g_{N,p,\alpha}(z)\}|^2\} \leq \frac{C(z,p,\alpha,X_2)}{N^{1/2}}, \tag{2.12}$$

(*ii*) *there exists the limiting probability measure $\sigma_{p,\alpha}$ such that*

$$g_{\sigma_{p,\alpha}}(z) = \lim_{N\to\infty} \mathbb{E}\{g_{N,p,\alpha}(z)\} = -ih(iz), \tag{2.13}$$

for an arbitrary compact in $\mathbb{C}_+$, the convergence is uniform, and the function $h(z) : \mathbb{C}_+ \to \mathbb{C}_+$ can be expressed in terms of the pair of functions $f_1(u,z)$ and $f_2(u,z)$ (*see* (2.7)–(2.8)), *which is a unique solution of the system of functional equations* (2.9) *in the class $\mathcal{F}$.*

3. Proof of Theorem 1

For any z: $\Re z > 0$, consider the functions $f_{1,N}(u,z) : \mathbb{R}_+ \to \mathbb{C}$, $f_{2,N}(u,z) : \mathbb{R}_+ \to \mathbb{C}$:

$$f_{1,N}(u,z) = \frac{1}{[\alpha N]} \sum_{k=1}^{[\alpha N]} e^{-ua_k^2 G_{kk}^{(N,p,\alpha)}(z)}, \quad G^{(N,p,\alpha)}(z) = (z - iA^{(N,p,\alpha)})^{-1}, \tag{3.1}$$

$$f_{2,N}(u,z) = \frac{1}{N - [\alpha N]} \sum_{k=[\alpha N]+1}^{N} e^{-ua_k^2 G_{kk}^{(N,p,\alpha)}(z)},$$

where $\{a_i\}_{i=1}^{\infty}$ is a family of i.i.d. random variables which do not depend on $\{a_{i,j}\}_{i<j}^{\infty}$ and have the same probability distribution as $a_{1,2}$. It is easy to see that $G_{NN}^{(N,p,\alpha)}(z)$ can be represented in the form

$$G_{NN}^{(N,p)}(z) = \left(z - iA_{NN}^{(N,p,\alpha)} + \sum_{j,k=1}^{N-1} \widetilde{G}_{jk}^{(N-1,p,\alpha)} A_{Nj}^{(N,p,\alpha)} A_{Nk}^{(N,p,\alpha)}\right)^{-1}, \tag{3.2}$$

where the matrix $\{\widetilde{G}_{ij}^{(N-1,p,\alpha)}(z)\}_{i,j=2}^{N}$ is a resolvent of the matrix $i\widetilde{A}^{(N-1,p,\alpha)}$, which can be obtained from $A^{(N,p,\alpha)}$ by deleting the last column and the last row. Let us use the formula (see [1]):

$$e^{-ua^2R} = 1 - u^{1/2}|a|\int\limits_0^\infty dv\frac{\mathcal{J}_1(2|a|\sqrt{uv})}{\sqrt{v}}\exp\{-R^{-1}v\}, \tag{3.3}$$

which is valid for any $u \geq 0$, $\Re R > 0$. Then, on the basis of (3.2), we get

$$\begin{aligned}\exp\{-ua_N^2G_{NN}^{(N,p,\alpha)}\} &= 1 - u^{1/2}|a_N|\int\limits_0^\infty dv\frac{\mathcal{J}_1(2|a_N|\sqrt{uv})}{\sqrt{v}} \\ &\qquad \exp\{-zv - v\sum_{j,k=1}^{[\alpha N]}\widetilde{G}_{ij}^{(N-1,p,\alpha)}A_{Ni}^{(N,p,\alpha)}A_{Nj}^{(N,p,\alpha)}\}.\end{aligned} \tag{3.4}$$

Denote

$$R_N(z) = \sum_{j,k=1,j\neq k}^{[\alpha N]}\widetilde{G}_{jk}^{(N-1,p)}A_{Nj}^{(N,p,\alpha)}A_{Nk}^{(N,p,\alpha)}. \tag{3.5}$$

Proposition 2.

$$\mathbb{E}\{|R_1(z)|^2\} \leq 2\frac{p^2X_2^2}{N|\Re z|^2} + \frac{p^4X_1^4}{N^2|\Re z|^2} + 4\frac{p^3X_1^2X_2}{N^2|\Re z|^2} + 6\frac{p^4X_1^4}{N|\Re z|^2} + 8\frac{p^3X_1^2X_2}{N|\Re z|^2}, \tag{3.6}$$

where

$$X_k = \int |a|^k d\mu(a).$$

P r o o f.

$$\begin{aligned}\mathbb{E}\{|R_1(z)|^2\} = &\sum_{\neq\{j_1,j_2,k_1,k_2\}}^{[\alpha N]}\mathbb{E}\Big\{\widetilde{G}_{j_1k_1}^{(N-1,p)}\overline{\widetilde{G}_{j_2k_1}^{(N-1,p)}}A_{Nj_1}^{(N,p)}A_{Nj_2}^{(N,p)}A_{Nk_1}^{(N,p)}A_{Nk_2}^{(N,p)}\Big\} \\ &+ 4\sum_{\neq\{j,k_1,k_2\}}^{[\alpha N]}\mathbb{E}\Big\{\widetilde{G}_{jk_1}^{(N-1,p)}\overline{\widetilde{G}_{jk_1}^{(N-1,p)}}|A_{Nj}^{(N,p)}|^2A_{Nk_1}^{(N,p)}A_{Nk_2}^{(N,p)}\Big\} \\ &+ 2\sum_{j\neq k}^{[\alpha N]}\mathbb{E}\Big\{\widetilde{G}_{jk}^{(N-1,p)}\overline{\widetilde{G}_{jk}^{(N-1,p)}}|A_{Nj}^{(N,p)}|^2|A_{Nk}^{(N,p)}|^2\Big\} = H_1 + 4H_2 + 2H_3.\end{aligned} \tag{3.7}$$

Averaging with respect to $\{A_{N,i}^{(N,p)}\}_{i=1}^{N-1}$ and using the fact that $\{\widetilde{G}_{ij}^{(N-1,p)}(z)\}_{i,j=1}^{N-1}$ do not depend on $A_{N,i}^{(N,p)}$, we obtain

$$\begin{aligned}
H_1 &\le X_1^4\frac{p^2}{N^2}\mathbb{E}\left\{\left|N^{-1}\sum_{j,k}^{[\alpha N]}\widehat{G}_{jk}\right|^2\right\}+6\frac{p^4X_1^4}{N|\Re z|^2}\le\frac{p^2X_1^4}{N^2|\Re z|^2}+6\frac{p^4X_1^4}{N|\Re z|^2},\\
H_2 &\le X_1^2X_2\frac{p}{N^3}\sum_{k_1\neq k_2}^{[\alpha N]}\mathbb{E}\left\{[\widehat{G}\widehat{G}^*]_{k_1k_2}\right\}+2\frac{p^3X_1^2X_2}{N|\Re z|^2}\le\frac{pX_1^2X_2}{N^2|\Re z|^2}+2\frac{p^3X_1^2X_2}{N|\Re z|^2},\\
H_3 &\le \frac{X_2^2}{N^2}\sum_{k}^{[\alpha N]}\mathbb{E}\left\{[\widehat{G}\widehat{G}^*]_{kk}\right\}\le\frac{X_2^2}{N|\Re z|^2}.
\end{aligned}$$

Besides, since evidently that

$$\Re\left\{\sum\widetilde{G}_{ij}^{(N-1,p)}A_{Ni}^{(N,p,\alpha)}A_{Nj}^{(N,p,\alpha)}\right\}\ge 0,\quad \Re\left\{\sum\widetilde{G}_{jj}^{(N-1,p,\alpha)}|A_{jj}^{(N,p,\alpha)}|^2\right\}\ge 0,$$

the inequality

$$|e^{-z_1}-e^{-z_2}|\le|z_1-z_2|\quad(\Re z_1,\Re z_2\ge 0)\tag{3.8}$$

and (3.4) imply

$$\begin{aligned}
\mathbb{E}\exp\{-ua_N^2G_{NN}^{(N,p,\alpha)}\} &= 1-u^{1/2}\mathbb{E}|a_N|\int_0^\infty dv\frac{\mathcal{J}_1(2|a_N|\sqrt{uv})}{\sqrt{v}}\\
&\quad\times\mathbb{E}_2\exp\{-zv-v\sum_{i=1}^{[\alpha N]}\widetilde{G}_{ii}^{(N-1,p,\alpha)}|A_{Nj}^{(N,p,\alpha)}|^2\}+\widetilde{r}_N(u),
\end{aligned}\tag{3.9}$$

where $\mathbb{E}_2$ denotes the averaging over $\{a_{ij}\}_{i,j}$ and $\left\{d_{ij}^{(N,p)}\right\}_{i,j}$. Remainder $\widetilde{r}_N(u)$ obeys the following condition:

$$|\widetilde{r}_N(u)|\le\mathbb{E}|R_1|u^{1/2}\mathbb{E}|a_N|\int_0^\infty dv|\frac{\mathcal{J}_1(2|a_1|\sqrt{uv})}{\sqrt{v}}e^{-zv}|\le C|R_1|u^{1/2}\mathbb{E}|a_N||\Re z|^{-1/2}.$$

In the last inequality we use the estimate for the Bessel function

$$|\mathcal{J}_1(u)|\le 1.\tag{3.10}$$

Here and below we denote by C some constants (different in different formulas), which do not depend on N,z,p,α. Taking into account (3.6), we get

$$|\widetilde{r}_N(u)|\le\frac{C(p,X_2)u^{1/2}}{\sqrt{N}|\Re z|^{5/2}}.\tag{3.11}$$

Since $\widetilde{G}^{(N-1,p)}(z)$ does not depend on $\left\{d_{N,j}^{(N,p)}\right\}_{j=1}^{N}$, we obtain

$$\mathbb{E}_2 \exp\{-v\sum_{i=1}^{[\alpha N]}\widetilde{G}_{ii}^{(N-1,p,\alpha)}|A_{Nj}^{(N,p,\alpha)}|^2\} = \mathbb{E}\Big\{\prod_{j=1}^{[\alpha N]}\Big((1-\frac{p}{N})+\frac{p}{N}e^{-va_{Nj}^2\widetilde{G}_{jj}^{(N-1,p)}}\Big)\Big\}$$
$$= e^{-\alpha p}\mathbb{E}\Big\{\exp\{\alpha p\widetilde{f}_{1,N-1}(v,z)\}\Big\} + R_N(v), \tag{3.12}$$

where

$$\widetilde{f}_{1,N-1}(u,z) = \frac{1}{[\alpha N]}\sum_{k=1}^{[\alpha N]} e^{-va_{Nj}^2\widetilde{G}_{kk}^{(N,p,\alpha)}(z)},$$

and R_N satisfies the estimate

$$|R_N(v)| \le \frac{Cp^2}{N}.$$

Using (3.8), we get

$$\Big|\mathbb{E}\exp\Big\{\alpha p\widetilde{f}_{1,N-1}(v,z)\Big\} - \exp\Big\{\alpha p\mathbb{E}\widetilde{f}_{1,N-1}(v,z)\Big\}\Big| \le$$
$$\le \alpha p e^{\alpha p}\mathbb{E}\Big|\widetilde{f}_{1,N-1}(v,z) - \mathbb{E}\widetilde{f}_{1,N-1}(v,z)\Big|. \tag{3.13}$$

Further considerations are based on Lemma 1.

Lemma 1. *Fix $\alpha \in (0,1)$. Let $A^{(n)}$ be a real symmetric $n\times n$ matrix such that*

$$A_{ij}^{(n)} = \begin{cases} a_{ij}, & \text{if } (i\in I_{\alpha,n}\wedge j\notin I_{\alpha,n})\vee(i\notin I_{\alpha,n}\wedge j\in I_{\alpha,n}),\\ 0, & \text{otherwise}\end{cases} \tag{3.14}$$

where $\{a_{ij}\}_{1\le i\le j}$ is a family of jointly independent identically distributed random variables that obey the following conditions:

$$\mathbb{E}|a_{ij}| \le \frac{C}{n},\ \mathbb{E}a_{ij}^2 \le \frac{C}{n}. \tag{3.15}$$

For z: $\Re z > 0$, consider

$$R = (z - iA)^{-1}, \quad F_n(z) = [\alpha n]^{-1}\sum_{j=1}^{[\alpha\cdot n]}\varphi_j(R_{jj}), \tag{3.16}$$

where random functions φ_j satisfy the condition

$$|\varphi_j'(\zeta)| \le C_3\,\alpha_j. \tag{3.17}$$

where $\{\alpha_j\}$ is a set of jointly independent identically distributed random variables also independent of $\{a_{ij}\}$ such that $\mathbb{E}\left\{\alpha_1^2\right\} < \infty$. Then

$$VarF_n(z) \leq \frac{4(1-\alpha)C_3^2}{\alpha n|\Re z|^4}\mathbb{E}\left\{\alpha_1^2\right\}\left(n\mathbb{E}|a_{12}| + (n\mathbb{E}a_{12}^2)^2\right). \tag{3.18}$$

The proof of Lemma 1 is given in Section 4.

R e m a r k 1. Lemma 1 is still valid for $F_n(z) = n^{-1}\sum_{j=1}^{n}\varphi_j(R_{jj})$ with changed constants.

Lemma 1 for $\varphi(\zeta) = \exp\{-va_{Nj}^2\zeta\}$, $\alpha_j = va_{Nj}^2$, $C_3 = 1$, $n = N-1$ implies

$$\mathbb{E}\left|\widetilde{f}_{1,N-1}(v,z) - \mathbb{E}\widetilde{f}_{1,N-1}(v,z)\right|^2 \leq \frac{\widetilde{C}^2(X_4,p)v^2}{\alpha N\left|\Re z\right|^4}. \tag{3.19}$$

Relations (3.13) and (3.19) yield

$$\left|\mathbb{E}\exp\left\{\alpha p\widetilde{f}_{1,N-1}(v,z)\right\} - \exp\left\{\alpha p\mathbb{E}\widetilde{f}_{1,N-1}(v,z)\right\}\right| \leq \alpha p e^{\alpha p}\frac{\widetilde{C}(X_4,p)v}{\alpha^{1/2}N^{1/2}\left|\Re z\right|^2}. \tag{3.20}$$

Combining (3.4)–(3.20), we get

$$\mathbb{E}f_{2,N}(u,z) = 1 - u^{1/2}e^{-\alpha p}\int\limits_0^\infty dve^{-zv}\frac{\mathcal{J}_1(2\sqrt{uv})}{\sqrt{v}}e^{\alpha p\mathbb{E}\widetilde{f}_{1,N-1}(v,z)} + r(u), \tag{3.21}$$

$$r(u) \leq \frac{\widetilde{C}(\mathbb{E}a_{Nj}^4,p)u^{1/2}}{N^{1/2}\left|\Re z\right|^{7/2}}.$$

In order to get the closed system of equations, we have to replace $\widetilde{f}_{1,N-1}$ by $f_{1,N}$. For this purpose we use the next bound on their difference

$$\left|\mathbb{E}f_{1,N}(v,z) - \mathbb{E}\widetilde{f}_{1,N-1}(v,z)\right| \leq \frac{\widetilde{C}(X_4,p)v}{\alpha N^{1/2}\left|\Re z\right|^2}. \tag{3.22}$$

Indeed, using Lemma 1 for $\varphi(\zeta) = \exp\{-va_j^2\zeta\}$, $\alpha_j = va_j^2$, $C_3 = 1$, $n = N$, we get

$$\mathbb{E}\left|f_{1,N}(v,z) - \mathbb{E}f_{1,N}(v,z)\right|^2 \leq \frac{\widetilde{C}^2(X_4,p)v^2}{\alpha N\left|\Re z\right|^4}. \tag{3.23}$$

Combining (3.23), (3.19) and (4.6) for $\varphi(\zeta) = \exp\{-va_{Nj}^2\zeta\}$, $\alpha_j = va_{Nj}^2$, $C_3 = 1$, $n = N$, we obtain (3.22).

The inequalities (3.22), (3.8) and (3.21) imply

$$\mathbb{E}f_{2,N}(u,z) = 1 - u^{1/2}e^{-\alpha p}\int_0^\infty dv e^{-zv}\frac{\mathcal{J}_1(2\sqrt{uv})}{\sqrt{v}}e^{\alpha p \mathbb{E}f_{1,N}(v,z)} + r(u), \tag{3.24}$$

$$r(u) \le \frac{\widetilde{C}(X_4,p)u^{1/2}}{N^{1/2}\,|\Re z|^{7/2}}.$$

Let us consider the Banach space $\mathcal{H}$ of all the functions $h : \mathbb{R}_+ \to \mathbb{C}$ which possess the norm (2.3). The space $\mathcal{H}^2$ possesses the norm $||(h_1,h_2)||_{\mathcal{H}^2} = \max\{||h_1||_{\mathcal{H}}, ||h_2||_{\mathcal{H}}\}$.

Define the operator $F_z : \mathcal{H}^2 \to \mathcal{H}^2$ of the form

$$F_z(\varphi_1,\varphi_2) = (\psi_1,\psi_2),\ \ \psi_1(u) = L(f_2,\mu,1-\alpha),\ \ \psi_2(u) = L(f_1,\mu,\alpha). \tag{3.25}$$

Let us denote by $B_{0,2} = \{h \in \mathcal{H}^2, \|h\|_{\mathcal{H}^2} \le 2\}$ the ball of radius 2 centered in the origin. Then for any φ_1, φ_2 : $||\varphi_1|| \le 2$, $\|\varphi_2\| \le 2$,

$$\|F_z(\varphi_1) - F_z(\varphi_2)\| \le X_1 p e^{2p+\frac{2p^2}{|\Re z|}}\left(\frac{2}{|\Re z|} + \frac{4}{|\Re z|^{1/2}}\right)\|\varphi_1-\varphi_2\|. \tag{3.26}$$

Indeed, inequalities (3.8), (3.10) imply

$$\|F_z(\varphi_1) - F_z(\varphi_2)\|_{\mathcal{H}^2} \le X_1 p\|\varphi_1-\varphi_2\|\int_0^\infty \frac{dv(1+v^{1/2})e^{-|\Re z|v+2p(1+v^{1/2})}}{\sqrt{v}}. \tag{3.27}$$

Using the trivial inequality

$$2pv^{1/2} - v|\Re z|/2 \le 2\frac{p^2}{|\Re z|}, \tag{3.28}$$

we get (3.26).

It is easy to see that $\|F_z(0)\|$ obeys the inequality

$$\|F_z(0)\|_{\mathcal{H}^2} \le \sup_{u>0}\frac{1+2X_1|\Re z|^{-1/2}u^{1/2}}{1+u^{1/2}} \le 1 + 2X_1|\Re z|^{-1/2}. \tag{3.29}$$

Thus there is $M > 0$ such that

$$\|F_z(\varphi_1) - F_z(\varphi_2)\| < 1/4\|\varphi_1-\varphi_2\|, \quad \|F_z(0)\| \le \frac{5}{4}, \quad z \in L(M), \tag{3.30}$$

where $L(M) = \{z : |\Re z| > M\}$.

Therefore, $F_z : B_{0,2} \to B_{0,2}$, and $F_z|_{B_{0,2}}$ is a contraction mapping for all $z \in L(M)$. Hence there exists the unique fixed point $f(u,z)$ which is a solution of (2.9). Since $|\mathbb{E}f_{1,N}(u,z)| \le 1, |\mathbb{E}f_{2,N}(u,z)| \le 1$, $\mathbb{E}f_N(u,z) \in B_{0,2}$, then the estimates (3.30) imply

$$\mathbb{E}f_N(u,z) = F_z(\mathbb{E}f_N(u,z)) + r_N(u,z) = \ldots = f(u,z) + r_N^{\infty}(u,z), \qquad (3.31)$$

where

$$\|r_N^{\infty}(u,z)\| \le \|r_N(u,z)\| \sum_{k=0}^{\infty} \frac{1}{4^k} \le \frac{4C(p,M)}{3N^{1/2}}, \quad z \in L(M). \qquad (3.32)$$

Hence, $\mathbb{E}f_N(u,z) \rightrightarrows f(u,z), \quad z \in L(M)$. Fix u. Since $\mathbb{E}f_{1,N}(u,z), \mathbb{E}f_{2,N}(u,z)$ are analytic and uniformly bounded for arbitrary $\Pi_{\varepsilon,a} = \{z : \varepsilon \le \Re z \le 2M, |\Im z| \le a\}$, by the Arzela theorem we can choose a subsequence $\{N_k\}_{k=1}^{\infty}$ such that $\mathbb{E}f_{1,N_k}(u,z) \rightrightarrows \widetilde{f}_1^{(a,\varepsilon)}(u,z)$, $\mathbb{E}f_{2,N_k}(u,z) \rightrightarrows \widetilde{f}_2^{(a,\varepsilon)}(u,z)$ in $\Pi_{\varepsilon,a}$. Then $f_1^{(a,\varepsilon)}(u,z)$, $f_2^{(a,\varepsilon)}(u,z)$ are analytic in $\Pi_{\varepsilon,a}$. But

$$f_2^{(a,\varepsilon)}(u,z) = f_2(u,z),\ f_1^{(a,\varepsilon)}(u,z) = f_1(u,z), \quad |\Re z| > M.$$

The uniqueness theorem of complex analysis and the arbitrariness of choosing the subsequence imply the existence of the analytic extension of $f_1(u,z)$ ($f_2(u,z)$ in $\mathbb{C}_+$ and the uniform convergence in z $f_{\alpha,N}(u,z) \rightrightarrows f_\alpha(u,z)$ for an arbitrary compact in $\mathbb{C}_+$. Thus, if we fixed any $z : \Re z > 0$, we obtain that $f_{\alpha,N}(u,z), (\alpha = 1,2)$ as a function of u converges pointwise to $f_{\alpha,N}(u,z)$. But since $|\frac{\partial}{\partial u} f_{\alpha,N}(u,z)| \le C$ and $|f_{\alpha,N}(u,z)| \le 1$, the pointwise convergence imply also the convergence in the norm (2.3). Then, using Lebesgue's dominated convergence theorem, we can prove that $f(u,z) = F_z(f)(u,z)$ in $\mathbb{C}_+$.

Indeed, using Lemma 1 for $\varphi(\zeta) = \zeta$, $\alpha_j = 1$, $C_3 = 1$, $n = N$, we get (2.12).

Uniform convergence $f_N(u,z)$ in $u \in [0,1]$ imply

$$\mathbb{E}g_{N,p,\alpha} = -i(\mathbb{E}a_1^2)^{-1} \frac{1}{N} \sum_{k=1}^{N} \mathbb{E}a_k^2 \mathbb{E}\{G_{kk}^{(N,p,\alpha)}\} =$$
$$= i(\mathbb{E}a_1^2)^{-1} \left(\alpha \frac{\partial}{\partial u} \mathbb{E}f_{1,N}(u,z)\Big|_{u=0} + (1-\alpha) \frac{\partial}{\partial u} \mathbb{E}f_{2,N}(u,z)\Big|_{u=0} \right). \qquad (3.33)$$

The next simple proposition allows us to make a final step.

Proposition 3. *Set* $\Psi_n(u) = \dfrac{f_n(u) - f_n(0)}{u} - f_n'(0)$. *Assume that*

$$|\Psi_n(u)| \le \varepsilon(u),\ \varepsilon(u) \to 0,\ as\ u \to 0. \qquad (3.34)$$

If there exists $f(u) = \lim_{n\to\infty} f_n(u)$ *and the function* f *is differential at* $u = 0$*, then*

$$\lim_{n\to\infty} f_n'(0) = f'(0). \tag{3.35}$$

P r o o f.

$$\Big|\frac{f_{1,n}(u,z) - f_{1,n}(0,z)}{u} - \frac{\partial}{\partial u} f_{1,n}(u,z)\Big|_{u=0}\Big|$$

$$= \Big|\int_0^1 \Big(\frac{\partial}{\partial u} f_{1,n}(tu,z) - \frac{\partial}{\partial u} f_{1,n}(u,z)\Big|_{u=0}\Big) dt\Big|$$

$$\le \frac{1}{[\alpha N]} \sum_{k=1}^{[\alpha N]} \int_0^1 a_k^2 \Big|G_{kk}^{(N,p,\alpha)}\Big| \Big|e^{-uta_k^2 G_{kk}^{(N,p,\alpha)}} - 1\Big| dt d\mu(a_k)$$

$$\le \frac{1}{|\Re z|} \frac{1}{[\alpha N]} \sum_{k=1}^{[\alpha N]} \Bigg(\int_{|a^2| > u^{-1/2}} + \int_{|a^2| \le u^{-1/2}}\Bigg) \int a^2 \Big|e^{-uta_k^2 G_{kk}^{(N,p,\alpha)}} - 1\Big| d\mu(a_k) dt$$

$$\le \frac{1}{|\Re z|} \Bigg(2 \int_{|a^2| > u^{-1/2}} a^2 d\mu(a) + \frac{1}{|\Re z|} \sqrt{u} \int a^2 d\mu(a)\Bigg) \xrightarrow{u \to 0} 0. \tag{3.36}$$

The similar estimate with $f_{2,n}$ is valid too. Hence,

$$g_{p,\alpha}(z) = \lim_{N\to\infty} \mathbb{E} g_{N,p,\alpha}(z) = i(\mathbb{E}a_1^2)^{-1} \Big(\alpha \frac{\partial}{\partial u} f_1(u,z)\Big|_{u=0} + (1-\alpha) \frac{\partial}{\partial u} f_2(u,z)\Big|_{u=0}\Big). \tag{3.37}$$

4. Proofs of Auxiliary Statements

P r o o f of Lemma 1. Let us denote by $\mathbb{E}_k$ the averaging over $\{a_{ij}\}_{i \le j}$ $i \le k$ ($\mathbb{E}_n = \mathbb{E}$, $\mathbb{E}_0$ means the absence of averaging), by $n_1 = [\alpha n]$, by $n_2 = n - n_1$. Then

$$F_n - \mathbb{E}F_n = \sum_{k=0}^{n_1-1} (\mathbb{E}_k F_n - \mathbb{E}_{k+1} F_n) \Rightarrow \qquad \mathbb{E}|F_n - \mathbb{E}F_n|^2$$

$$= 2 \sum_{j<k}^{n_1-1} \mathbb{E}(\mathbb{E}_k F_n - \mathbb{E}_{k+1} F_n)(\mathbb{E}_j \overline{F}_n - \mathbb{E}_{j+1} \overline{F}_n) + \sum_{k=0}^{n_1-1} \mathbb{E}|\mathbb{E}_k F_n - \mathbb{E}_{k+1} F_n|^2$$

$$= \sum_{k=0}^{n_1-1} \mathbb{E}|\mathbb{E}_k F_n - \mathbb{E}_{k+1} F_n|^2. \tag{4.1}$$

Here we use the identity $\mathbb{E}(\mathbb{E}_kF_n-\mathbb{E}_{k+1}F_n)(\mathbb{E}_j\overline{F}_n-\mathbb{E}_{j+1}\overline{F}_n)=0$ for $j\neq k$.

Denote by $\mathbb{E}^{(k)}$ the averaging over $\{a_{kj}\}_{j=k}^n$. Let $F_n^{(k)}=F_n\Big|_{\{a_{kj}=0\}_{j=1}^n}$. So we get

$$\mathbb{E}|\mathbb{E}_kF_n-\mathbb{E}_{k+1}F_n|^2=\mathbb{E}|\mathbb{E}_k(F_n-\mathbb{E}^{(k+1)}F_n)|^2\leq\mathbb{E}\left|\mathbb{E}_k\left(F_n-F_n^{(k+1)}\right)\right|^2.$$

Taking into account the Schwarz inequality, we obtain $\mathbb{E}\left|\mathbb{E}_k\left(F_n-F_n^{(k)}\right)\right|^2\leq\mathbb{E}\mathbb{E}_k\left|\left(F_n-F_n^{(k+1)}\right)\right|^2=\mathbb{E}\left|F_n-F_n^{(k+1)}\right|^2$. Due to the symmetry, for all k, we have

$$\mathbb{E}|\mathbb{E}_kF_n-\mathbb{E}_{k+1}F_n|^2\leq\mathbb{E}\left|F_n-F_n^{(k+1)}\right|^2=\mathbb{E}\left|F_n-F_n^{(1)}\right|^2. \tag{4.2}$$

In order to estimate $\mathbb{E}\left|F_n-F_n^{(1)}\right|^2$, we introduce the matrix $A(t)$:

$$A_{ij}(t)=\left(\mathbf{1}_{i\in I_{\alpha,N},j\notin I_{\alpha,N}}+\mathbf{1}_{i\notin I_{\alpha,N},j\in I_{\alpha,N}}\right)\left(\mathbf{1}_{i\geq1,j\geq1}a_{ij}+t(1-\mathbf{1}_{i\geq1,j\geq1})a_{ij}\right).$$

Also, we introduce the functions

$$R(t)=(z-iA(t))^{-1},\quad F_n(t)=n_1^{-1}\sum_{j=1}^{n_1}\varphi(R_{jj}(t)).$$

Clearly, the equality

$$F_n-F_n^{(1)}=\int_0^1\frac{d}{dt}F_n(t)dt \tag{4.3}$$

is true. We can estimate $\frac{d}{dt}F(t)$ by the following way:

$$\frac{d}{dt}F_n(t)=\frac{1}{n_1}\sum_{j=1}^{n_1}\sum_{k,l}\xi_j(t)R_{jk}(t)A'_{kl}(t)R_{lj}(t)=\frac{2}{n_1}\sum_{j\in I_{\alpha,n},k,l}\xi_j(t)R_{j1}(t)a_{1k}(t)R_{kj}(t)=2H,$$

where

$$\xi_j(t)=\frac{\partial}{\partial R_{jj}}\varphi(R_{jj}(t)),$$

$$\mathbb{E}\left\{|H|^2\right\}\leq\frac{C_3^2}{4n_1^2}\mathbb{E}\left\{\sum_{j,j'\in I_{\alpha,n},k,k'}\alpha_j\alpha'_j\left(|R_{jk}|^2+|R_{1j}|^2\right)|a_{1k}|\left(\left|R_{j'k'}\right|^2+\left|R_{1j'}\right|^2\right)|a_{1k'}|\right\}$$
$$\leq\frac{C_3^2}{n_1^2}\mathbb{E}\left\{\alpha_1^2\right\}\mathbb{E}\left(\sum_k|a_{1k}|^2\right)\leq\frac{C_3^2}{n_1^2|\Re z|^4}\mathbb{E}\left\{\alpha_1^2\right\}\left(n_2\mathbb{E}|a_{12}|+(n_2\mathbb{E}a_{12}^2)^2\right). \tag{4.4}$$

Here we use the inequality

$$\sum_j |R_{sj}|^2 = [RR^*]_{ss} \le \frac{1}{|\Re z|^2}.$$

Using (4.3) and (4.4), we obtain

$$\mathbb{E}\left|F_n - F_n^{(1)}\right|^2 = \mathbb{E}\left\{\left|\int_0^1 \frac{d}{dt}F_n(t)\right|^2\right\} \le \int_0^1 \mathbb{E}\left\{\left|\frac{d}{dt}F_n(t)\right|^2\right\}$$
$$\le \frac{4C_3^2}{n_1^2|\Re z|^4}\mathbb{E}\left\{\alpha_1^2\right\}\left(n_2\mathbb{E}|a_{12}| + (n_2\mathbb{E}a_{12}^2)^2\right). \quad (4.5)$$

In much the same way the following estimate can be proved:

$$\mathbb{E}\left|F_n - F_n^{(n)}\right|^2 = \mathbb{E}\left\{\left|\int_0^1 \frac{d}{dt}F_n(t)\right|^2\right\} \le \int_0^1 \mathbb{E}\left\{\left|\frac{d}{dt}F_n(t)\right|^2\right\}$$
$$\le \frac{4C_3^2}{n_1^2|\Re z|^4}\mathbb{E}\left\{\alpha_1^2\right\}\left(n_2\mathbb{E}|a_{12}| + (n_2\mathbb{E}a_{12}^2)^2\right). \quad (4.6)$$

Combining (4.1), (4.2), (4.5), we obtain (3.18).

P r o o f of Proposition 1. Denote by $a^{(T)}$ the truncation of a with the parameter T, i.e.,

$$a^{(T)}(\omega) = \begin{cases} a(\omega), & \text{if } a(\omega) < T, \\ T, & \text{otherwise} \end{cases}. \quad (4.7)$$

Here and below the notation with an upper index (T) means that the function is defined for the matrix $A^{(T)}$ by the same way as it was done for A.

Similarly to (3.27) and (3.28), we can obtain that for any φ: $||\varphi|| \le 2$,

$$\|F_z(\varphi) - F_z^{(T)}(\varphi)\| \le \frac{2e^{2p+\frac{2p^2}{|\Re z|}}}{|\Re z|}\|\varphi\|\int_T^\infty |a|d\mu(a). \quad (4.8)$$

Combining (4.8) and (3.30), we obtain

$$\forall z \in L(M)\ f^{(T)}(u,z) \xrightarrow[T\to\infty]{\|\|_{\mathcal{H}^2}} f(u,z). \quad (4.9)$$

Theorem 2 yields

$$g_{\sigma^{(T)}_{p,\alpha}}(z) = \lim_{N\to\infty} \mathbb{E}\{g^{(T)}_{N,p,\alpha}(z)\}, \qquad (4.10)$$

and for an arbitrary compact in $\mathbb{C}_+$ the convergence is uniform.

Taking into account the resolvent identity and the Schwarz inequality, we obtain

$$\left|\mathbb{E}g_{\sigma^{(T)}_{N,p,\alpha}}(z) - \mathbb{E}g_{\sigma_{N,p,\alpha}}(z)\right| \le \frac{p}{N|\Im z|^2} \left(\int_T^\infty a^2 d\mu(a)\right)^{1/2}. \qquad (4.11)$$

Combining (4.9)–(4.11), we obtain

$$g_{\sigma_{p,\alpha}}(z) = \lim_{N\to\infty} \mathbb{E}\{g_{N,p,\alpha}(z)\}, \qquad (4.12)$$

and for an arbitrary compact in $\mathbb{C}_+$ the convergence is uniform.

(2.12) is still valid because it does not require the existence of X_4 (just use Lemma 1 for $\varphi(\zeta) = \zeta$, $\alpha_j = 1$, $C_3 = 1$, $n = N$).

References

[1] *M. Abramowitz and I. Stegun*, Handbook of Mathematical Functions. Dover, New York, 1972.

[2] *M. Bauer and O. Golinelli*, Random Incidence Matrices: Spectral Density at Zero Energy. Saclay preprint T00/087; cond-mat/0006472.

[3] *M. Bauer and O. Golinelli*, Random Incidence Matrices: Moments and Spectral Density. — *J. Stat. Phys.* **103** (2001), 301–336.

[4] *F. Benaych-Georges, A. Guionnet, and C. Male*, Central Limit Theorems for Linear Statistics of Heavy Tailed Random Matrices. — *Comm. Math. Phys.* **329** (2014), No. 2, 641–686.

[5] *B. Bollobas*, Random Graphs. New York, Acad. Press, 1985.

[6] *Y.V. Fyodorov and A.D. Mirlin*, Strong Eigenfunction Correlations near the Anderson Localization Transition. — *Phys. Rev. B* **55** (1997), R16001–R16004.

[7] *S. Janson, T. Łuczak, and A. Rucinski*, Random Graphs. John Wiley & Sons, Inc. New York, 2000.

[8] *O. Khorunzhy, M. Shcherbina, and V. Vengerovsky*, Eigenvalue Distribution of Large Weighted Random Graphs. — *J. Math. Phys.* **45** (2004), No, 4, 1648–1672.

[9] *O. Khorunzhy, B. Khoruzhenko, L. Pastur and M. Shcherbina*, The Large-n Limit in Statistical Mechanics and Spectral Theory of Disordered System. Phase Transition and Critical Phenomena. **15**, p. 73, Academic Press, 1992.

[10] *M.L. Mehta*, Random Matrices. New York, Acad. Press, 1991.

[11] *A.D. Mirlin and Y.V. Fyodorov*, Universality of the Level Correlation Function of Sparce Random Matrices. — *J. Phys. A: Math. Jen.* **24** (1991), 2273–2286.

[12] *G.J. Rodgers and A.J. Bray*, Density of States of a Sparse Random Matrix. — *Phys. Rev. B* **37** (1988), 3557–3562.

[13] *G.J. Rodgers and C. De Dominicis*, Density of States of Sparse Random Matrices. — *J. Phys. A: Math. Jen.* **23** (1990), 1567–1566.

[14] *M. Shcherbina and B. Tirozzi*, Central Limit Theorem for Fluctuations of Linear Eigenvalue Statistics of Large Random Graphs. — *J. Math. Phys.* **51** (2010), No. 2, 023523.

[15] *M. Shcherbina and B. Tirozzi*, Central Limit Theorem for Fluctuations of Linear Eigenvalue Statistics of Large Random Graphs: Diluted Regime. — *J. Math. Phys.* **53** (2012), No. 4, 043501.

[16] *V. Vengerovsky*, Eigenvalue Distribution of a Large Weighted Bipartite Random Graph. — *J. Math. Phys., Anal., Geom.* **10** (2014), No. 2, 240–255.

[17] *E.P. Wigner*, On the Distribution of the Roots of Certain Symmetric Matrices. — *Ann. Math.* **67** (1958), 325–327.