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On the Structure of Multidimensional Submanifolds with Metric of Revolution in Euclidean Space

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It is proved that a submanifold of low codimension with induced metric of revolution of sectional curvature of constant sign is a submanifold of revolution if the coordinate geodesic lines are the lines of curvature.

Key words: Metric of revolution, submanifolds of rotation, lines of curvature, sectional curvature.

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1. Introduction

The structure of surfaces of revolution with constant Gaussian curvature in the Euclidean space E^3 is well known. From the fact that the induced metric is a metric of revolution it does not follow that the surface in E^3 is a surface of revolution. The following example can be constructed by using the Cauchy–Kowalewski theorem. The analytic metric of revolution,

$$ds^2 = du^2 + \varphi^2(u)dv^2,$$

locally admits isometric embedding in E^3 such that the geodesic line $v = 0$ mapped onto a space curve with torsion is not equal to zero at any point. Therefore, it is naturally to ask when multidimensional submanifolds with induced metric of revolution are submanifolds of revolution. We give the answer to this question for submanifolds in Euclidean space. We distinguish 3 cases; namely, when extrinsic sectional curvature is 1) negative, 2) zero, 3) positive.

Definition 1.1. A multidimensional Riemannian metric on a manifold M^l is called a metric of revolution if there exists a regular coordinate system such that this Riemannian metric has the form

$$ds^2 = (du^1)^2 + \varphi^2(u^1)d\sigma^2, \tag{1.1}$$

where $\varphi(u^1)$ is a regular positive function, $d\sigma^2$ is the Riemannian metric of constant sectional curvature.

2. Submanifolds of negative sectional curvature in Euclidean space

Let F^l be a submanifold in the Euclidean space E^n such that the induced metric in some regular system of coordinates has the form

$$ds^2 = (du^1)^2 + \varphi^2(u^1)d\sigma^2,$$

where $d\sigma^2$ is a metric of constant curvature.

Definition 2.1. Let F^l be a submanifold in E^{l+p} and F^{l-1} be a submanifold of a unit sphere $S^{l+p-2} \subset E^{l+p-1}$ of constant extrinsic sectional curvature with the radius vector

$$\rho(u^2, \dots, u^l) = (\rho_1(u^2, \dots u^l), \dots, \rho_{l+p-1}(u^2, \dots u^l), 0).$$

Take a regular curve γ in the plane $E^2 = x^1Ox^{l+p}$:

$$x^1 = f(u^1), \quad x^{l+p} = h(u^1),$$

where u^1 is the arc-length parameter on the curve γ, $x^1, \dots, x^{l+p}$ is the orthogonal Cartesian coordinate system in E^{l+p}, O is the origin.

Then a submanifold $F^l \subset E^{l+p}$ with the radius vector

$$\begin{aligned} x^1 &= f(u^1)\rho_1(u^2, \dots, u^l); \\ &\dots \\ x^{l+p-1} &= f(u^1)\rho_{l+p-1}(u^2, \dots, u^l); \\ x^{l+p} &= h(u^1) \end{aligned}$$

is called a submanifold of revolution.

We say that the submanifold F^l is obtained by rotating the curve γ around the axis x^{l+p} along the submanifold F^{l-1}.

Example 2.2. The Shur submanifold F^l in E^{2l-1} with the radius vector

$$\begin{cases} x^1 = a^2 e^{-u^1} \cos\left(\dfrac{u^2}{a^2}\right); \\ x^2 = a^2 e^{-u^1} \sin\left(\dfrac{u^2}{a^2}\right); \\ \dots \\ x^{2l-3} = a^l e^{-u^1} \cos\left(\dfrac{u^l}{a^l}\right); \\ x^{2l-2} = a^l e^{-u^1} \sin\left(\dfrac{u^l}{a^l}\right); \\ x^{2l-1} = \displaystyle\int_0^{u^1} \sqrt{1 - e^{-2s}}ds \end{cases}$$

is a submanifold of revolution, where $i = 2, \dots, l$, $a^i \in R \setminus \{0\}$, $\sum\limits_{i=2}^{l} (a^i)^2 = 1$, $f(u^1) = e^{-u^1}$, $h(u^1) = \int_0^{u^1} \sqrt{1 - e^{-2s}} ds$.

The radius vector $x^1 = e^{-u^1}$, $x^{2l-1} = \int_0^{u^1} \sqrt{1 - e^{-2s}} ds$ defines the tractrix.

The Shur submanifold is obtained by rotating the tractrix along the torus F^{l-1} with the radius vector

$$\rho(u^2, \dots, u^l) = \left\{ a^2 \cos\left(\frac{u^2}{a^2}\right), a^2 \sin\left(\frac{u^2}{a^2}\right), \dots, a^l \cos\left(\frac{u^l}{a^l}\right), a^l \sin\left(\frac{u^l}{a^l}\right) \right\}.$$

At a submanifold of revolution, the metric of revolution is induced. The converse is true under the additional condition.

Definition 2.3. A line $\gamma \subset F^l \subset E^{l+p}$ is called a line of curvature of a submanifold F^l if for any normal ξ from the normal space NF^l the tangent vector $\dot\gamma$ is a principal direction of the second fundamental form with respect to the normal ξ.

The following theorem holds.

Theorem 2.4. *Suppose F^l is a C^3-regular submanifold in the Euclidean space E^{2l-1} with the induced metric of revolution of negative sectional curvature*

$$ds^2 = (du^1)^2 + \varphi^2(u^1) d\sigma^2,$$

where $d\sigma^2$ is a metric of constant curvature. If the coordinate lines u^1 are the lines of curvature of the submanifold F^l, then this submanifold is a submanifold of revolution.

Lemma 2.5. *Let F^l be a submanifold in the Euclidean space E^{2l-1} with the induced Riemannian metric of negative sectional curvature. Suppose the Riemannian metric has the form (1.1).*

1. *If $d\sigma^2$ is a flat metric, then $\varphi' \neq 0$, $\varphi'' \neq 0$, $\varphi > 0$;*
2. *If $d\sigma^2$ is the metric of a unit sphere, then $\varphi' > 1 (u^1 > 0)$, $\varphi'' > 0$, $\varphi(0) = 0$, $\varphi'(0) = 1$;*
3. *If $d\sigma^2$ is the metric of hyperbolic space of curvature -1, then $\varphi'' > 0$, $\varphi \geq 0$.*

Proof. 1. The matrix G of the coefficients of the first fundamental form of F^l has the form

$$G = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \varphi^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varphi^2 \end{pmatrix}. \tag{2.1}$$

Then the Christoffel symbols of the metric ds^2 have the form

$$\Gamma^j_{1i} = \delta^j_i \frac{\varphi'}{\varphi}, \quad \Gamma^1_{ij} = -\delta_{ij} \varphi \varphi', \qquad i, j = 2, \dots, l. \tag{2.2}$$

The sectional curvatures of F^l along the coordinate 2-dimensional planes π_{1j}, π_{ij} are

$$K(\pi_{1j}) = -\frac{\varphi''}{\varphi}, \quad K(\pi_{ij}) = -\frac{(\varphi')^2}{\varphi^2}. \tag{2.3}$$

Since the sectional curvatures of F^l are negative, we obtain $\varphi' > 0, \varphi'' > 0$.

2. By assumption, the metric of F^l is regular. From the singularity of polar coordinates it follows that $\varphi(0) = 0, \varphi'(0) = 1$. By direct computation, we get

$$K(\pi_{1j}) = -\frac{\varphi''}{\varphi} < 0, \quad K(\pi_{ij}) = \frac{1-(\varphi')^2}{\varphi^2} < 0. \tag{2.4}$$

From 2.4 we obtain $\varphi' > 1(u^1 > 0)$, $\varphi'' > 0$.

3. By assumption, the metric of F^l is regular. From this fact it follows that $\varphi > 0$. From direct computation, we get

$$K(\pi_{1j}) = -\frac{\varphi''}{\varphi} < 0, \quad K(\pi_{ij}) = \frac{-1-(\varphi')^2}{\varphi^2} < 0.$$

This completes the proof of Lemma 2.5. □

Let $r = r(u^1, \ldots, u^l)$ be the radius vector of a submanifold F^l in the Euclidean space. By r_i, denote $\frac{\partial r}{\partial u^i}$, by r_{ij}, denote $\frac{\partial^2 r}{\partial u^i \partial u^j}$, $i, j = 1, \ldots l$.

Lemma 2.6. *Let F^l be a submanifolds in the Euclidean space E^{2l-1} with the induced Riemannian metric of revolution of negative sectional curvature*

$$ds^2 = (du^1)^2 + \varphi^2(u^1)d\sigma^2.$$

If the coordinate lines u^1 are the lines of curvature, then the rank of the map

$$\tilde{r} = r - \frac{\varphi}{\varphi'} r_1$$

is equal to one.

Proof. 1. Suppose that $d\sigma^2 = (du^2)^2 + \ldots + (du^l)^2$. By

$$B^\sigma = \begin{pmatrix} b^\sigma_{11} & \ldots & b^\sigma_{1l} \\ \vdots & & \vdots \\ b^\sigma_{l1} & \ldots & b^\sigma_{ll} \end{pmatrix}, \qquad \sigma = 1, \ldots, l-1,$$

denote the matrices of the coefficients of the second fundamental forms F^l with respect to orthonormal basis of normals $\xi_1, \ldots, \xi_{l-1}$. From the condition that the coordinate lines are the lines of curvature it follows that

$$b^\sigma_{1i} = 0, \quad \sigma = 1, \ldots, l-1, \; i = 2, \ldots, l.$$

Now we calculate the Jacobi matrix of the map

$$\tilde{r} = r - \frac{\varphi}{\varphi'} r_1,$$

$$\tilde{r}_1 = \frac{\varphi\varphi''}{(\varphi')^2} r_1 - \frac{\varphi}{\varphi'} r_{11},$$
$$\tilde{r}_j = r_j - \frac{\varphi}{\varphi'} r_{1j}, \qquad j = \overline{2, l}.$$

Since

$$r_{ij} = \Gamma_{ij}^k r_k + b_{ij}^\sigma \xi_\sigma$$

and only the following Christoffel symbols are not equal to zero

$$\Gamma_{1j}^j = \frac{\varphi}{\varphi'}, \quad \Gamma_{jj}^1 = -\varphi\varphi',$$

we obtain

$$\tilde{r}_1 = \frac{\varphi\varphi''}{(\varphi')^2} r_1 + b_{11}^\sigma \xi_\sigma \neq 0, \quad \tilde{r}_j = b_{1j}^\sigma \xi_\sigma = 0. \tag{2.5}$$

Using (2.5), we get that the Jacobi matrix of the map $\tilde{r}$ is

$$J = \frac{\varphi}{\varphi'} \begin{pmatrix} \frac{\varphi''}{\varphi'} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_{11}^1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_{11}^{l-1} & 0 & \cdots & 0 \end{pmatrix}. \tag{2.6}$$

It now follows that the rank of the Jacobi matrix J is equal to one and the map $\tilde{r}$ depends only on the variable u^1, and we get $\tilde{r} = \Phi(u^1)$.

2. When $d\sigma^2$ is the metric of positive or negative constant curvature, the proof is similar. □

Proof of Theorem 2.4. For technical reasons, we devide the proof of the theorem into three parts.

1. Let $d\sigma^2$ be a flat metric, $d\sigma^2 = (du^2)^2 + \ldots + (du^l)^2$. Consider an ordinary differential equation

$$r - \frac{\varphi}{\varphi'} r_1 = \Phi(u^1)$$

with respect to the vector function r. Solving this equation, we get

$$r = -\varphi(u^1) \int_0^{u^1} \Phi(t) \frac{\varphi'(t)}{\varphi^2(t)} dt + \varphi(u^1)\rho(u^2, \ldots, u^l).$$

We set

$$\psi(u^1) = -\varphi(u^1) \int_0^{u^1} \Phi(t) \frac{\varphi'(t)}{\varphi^2(t)}\, dt.$$

Then we have

$$r = \psi(u^1) + \varphi(u^1)\rho(u^2, \ldots, u^l).$$

The vectors tangent to the coordinate lines of F^l have the form

$$r_1 = \psi'(u^1) + \varphi'(u^1)\rho(u^2, \dots, u^l),$$
$$r_j = \varphi(u^1)\rho_j(u^2, \dots, u^l).$$

Since the metric of F^l is written as

$$ds^2 = (du^1)^2 + \varphi^2(u^1)((du^2)^2 + \dots + (du^l)^2),$$

it follows that the coefficients of the first fundamental form are

$$g_{11} = \langle \psi', \psi' \rangle + 2\varphi' \langle \psi', \rho \rangle + (\varphi')^2 \langle \rho, \rho \rangle = 1, \tag{2.7}$$
$$g_{1j} = \varphi \langle \psi', \rho_j \rangle + \varphi\varphi' \langle \rho, \rho_j \rangle = 0, \tag{2.8}$$
$$g_{ij} = (\varphi)^2 \langle \rho_i, \rho_j \rangle = \delta_{ij}\varphi^2. \tag{2.9}$$

Take the origin of coordinates at the point $\Phi(0)$. Then for $u^1 = 0$,

$$r_1(0) = \varphi'(0)\rho(u^2, \dots, u^l),$$

and formula (2.7) transforms to

$$(\varphi'(0))^2 \langle \rho, \rho \rangle = 1$$

for any $u^2, \dots, u^l$. It follows that

$$\langle \rho, \rho \rangle = \frac{1}{(\varphi'(0))^2} \tag{2.10}$$

and the submanifold F^{l-1} with the radius vector $\rho = \rho(u^2, \dots, u^l)$ belongs to the sphere of radius $\frac{1}{\varphi'(0)}$. From (2.9), we get that the submanifold F^{l-1} has a flat metric.

Let us show that the submanifold F^{l-1} does not belong to the Euclidean space E^{2l-3}. In the converse case, F^{l-1} is a submanifold of the sphere $S^{2l-4} \subset E^{2l-3}$. It is known that

$$K_{\text{ext}} = K_{\text{int}} - K,$$

where $K = (\varphi'(0))^2$ is the curvature of the sphere, $K_{\text{int}} = 0$. It follows that $K_{\text{ext}} = -(\varphi'(0))^2$ and F^{l-1} is an intrinsic flat submanifold of extrinsic negative sectional curvature in the sphere S^{2l-4}. It is known that if a submanifold F^m of a Riemannian space M^{m+p} has negative extrinsic sectional curvature, then $p \geq m - 1$ [1, Theorem 3.2.2]. In our case $m = l - 1$, $M^{m+p} = S^{2l-4}$, $p = l - 3 = m - 2$. This contradiction concludes that the codimension of F^{l-1} is equal to $l-2$, $F^{l-1} \subset S^{2l-3}$. From (2.8), it follows that

$$\langle \psi', \rho_j \rangle + \varphi' \langle \rho, \rho_j \rangle = 0. \tag{2.11}$$

From (2.10) and (2.11), we obtain

$$\langle \psi', \rho_j \rangle = 0. \tag{2.12}$$

Differentiating (2.12) with respect to u^1, we get

$$\langle \psi'', \rho_j \rangle = 0, \ \langle \psi''', \rho_j \rangle = 0. \tag{2.13}$$

We compute the derivative of the function ψ,

$$\psi'(u^1) = -\varphi' \int_0^{u^1} \frac{\varphi'}{\varphi^2} \Phi(t) dt - \frac{\varphi'(u^1)}{\varphi} \Phi(u^1).$$

The values of the derivatives ψ'', ψ''' at the point $u^1 = 0$ are

$$\begin{aligned} \psi''(0) &= -\frac{\varphi'(0)}{\varphi(0)} \Phi'(0), \\ \psi'''(0) &= -\frac{(\varphi'(0))^2 - 2\varphi''(0)\varphi(0)}{\varphi^2(0)} \Phi'(0) - \frac{\varphi'(0)}{\varphi(0)} \Phi''(0). \end{aligned}$$

And equations (2.13) have the form

$$\langle \Phi'(0), \rho_j \rangle = 0, \quad \langle \Phi''(0), \rho_j \rangle = 0. \tag{2.14}$$

Let us prove that the vectors $\Phi'(0)$, $\Phi''(0)$ are collinear. Assume the contrary. Equations (2.14) are true for $\Phi'(u^1)$, $\Phi''(u^1)$. We can rewrite (2.14) in the following way:

$$\langle \Phi'(u^1), \rho \rangle = c_1(u^1), \quad \langle \Phi''(u^1), \rho \rangle = c_2(u^1). \tag{2.15}$$

From (2.15), it follows that the submanifold F^{l-1} belongs to the Euclidean space E^{2l-3}. We have proved before that it is impossible. The vectors $\Phi'(u^1)$, $\Phi''(u^1)$ are collinear for any point on the curve $\Phi(u^1)$. Thus the curve $\Phi(u^1)$ is a segment of a straight line and the submanifold F^{l-1} belongs to the Euclidean space E^{2l-2} orthogonal to the segment $\Phi(u^1)$. In E^{2l-1}, choose an orthogonal coordinate system such that the axis x^{2l-1} coincides with the straight line $\Phi(u^1)$. Then $\Phi(u^1) = (0, \dots, \mu(u^1))$. Hence the radius vector of the submanifold F^l has the form

$$\begin{cases} x^1 = \varphi(u^1)\rho^1(u^2, \dots, u^l); \\ \dots \\ x^{2l-2} = \varphi(u^1)\rho^{2l-2}(u^2, \dots, u^l); \\ x^{2l-1} = -\varphi(u^1) \displaystyle\int_0^{u^1} \mu(t) \frac{\varphi'(t)}{\varphi^2(t)} dt. \end{cases}$$

This is a submanifold of revolution with the meridian curve

$$\begin{cases} x^1 = \varphi(u^1); \\ x^{2l-1} = -\varphi(u^1) \displaystyle\int_0^{u^1} \mu(t) \frac{\varphi'(t)}{\varphi^2(t)} dt. \end{cases}$$

This completes the proof of Theorem 2.4, part 1.

2. Let $d\sigma^2$ be a metric of positive sectional curvature of the curvature one,

$$d\sigma^2 = \frac{4}{(1+(u^2)^2+\cdots+(u^l)^2)^2}((du^2)^2+\cdots+(du^l)^2).$$

In this case, $u^1 \geq u^1_0 > 0$. Assume that $u^1_0 = 0$. Every submanifold $u^1 = u^1_0$ belongs to the sphere of radius $\varphi'(u^1_0)$ with the center $\Phi(0)$. From Lemma 2.5, part 2, it follows that $\varphi'(u^1_0) > 1$, $\varphi''(u^1_0) > 0$. We obtain that the sphere of radius $\varphi'(0) = 1$ contains F^l inside and is the supporting sphere F^l at the point $u^1 = 0$. The normal to the sphere $\varphi'(0)$ is the normal to F^l at the point $u^1 = 0$. And the second fundamental form F^l with respect of the normal to the sphere $\varphi'(0)$ is positive definite. But any submanifold F^l of negative sectional curvature in E^{2l-1} has 2^{l-1} asymptotic directions at every point [2, Lemma 3.2.1]. This contradiction concludes that $u^1 > u^1_0 > 0$. From the condition $u^1 \geq u^1_0 > 0$, it follows that the extrinsic curvature of submanifold F^{l-1} with the radius vector $\rho = \rho(u^2,\dots,u^l)$, $\langle\rho,\rho\rangle = \frac{1}{(\varphi'(u^1_0))^2}$ has the form

$$K_{ext} = 1 - (\varphi'(u^1_0))^2 < 0.$$

By the same argument as in part 1, the curve $\Phi(u^1)$ is a line, and this completes the proof.

3. When $d\sigma^2$ is the metric of constant negative curvature, the proof is similar to that of part 1. □

3. Submanifolds with zero sectional curvature in Euclidean space

Let F^l be a hypersurface with the induced metric of revolution of zero sectional curvature in the Euclidean space E^{l+1}. In this case, the metric of the hypersurface has one of the following forms:

(I) $ds^2 = (du^1)^2 + (du^2)^2 + \ldots + (du^l)^2$,

(II) $ds^2 = (du^1)^2 + (u^1)^2 d\sigma^2$,

where $d\sigma^2$ is the metric of the unit sphere. But it can not be of the form

$$ds^2 = (du^1)^2 + \varphi^2(u^1)d\sigma^2,$$

where $d\sigma^2$ is the metric of the Lobachevsky space.

It is easy to compute that this metric is not a flat metric.

Theorem 3.1. *Suppose that F^l is a hypersurface of zero sectional curvature.*

1) *If the coordinate lines u^1 of metric (I) are the lines of curvature, then*

   a) *either F^l is a cylinder with the one-dimensional generator over a hypersurface F^{l-1} isometric to the Euclidean space E^{l-1} in the Euclidean space E^l,*

   b) *or F^l is a cylinder with the $(l-1)$-dimensional generator over a plane curve.*

2) *If the coordinate lines u^1 of metric (II) are the lines of curvature, then F^l is a cone over a local isometric immersion of a domain of the unit sphere S^{l-1} into the unit sphere $S^l \subset E^{l+1}$. The radius vector of the submanifold F^l has the form*

$$r = u^1\rho(u^2, \dots, u^l),$$

where ρ is the radius vector of the submanifold $F^{l-1} \subset S^l$ which is locally isometric to the unit sphere.

Proof. 1) From the conditions of the theorem and the Weingarten formulas, it follows that

$$r_{ij} = 0, j = 2, \dots, l.$$

From the above,

$$r = f(u^1) + \rho(u^2, \dots, u^l).$$

From the form of the metric, we get

$$\langle f', f'\rangle = \langle \rho_j, \rho_j\rangle = 1, \quad \langle \rho_i, \rho_j\rangle = \delta_{ij}, \quad \langle f', \rho_j\rangle = 0. \tag{3.1}$$

Differentiating the last equation, we obtain

$$\langle f'', \rho_j\rangle = 0.$$

Suppose that the vectors are non collinear for some interval u^1. Then F^{l-1} belongs to the Euclidean space E^{l-1} and is a domain in E^{l-1}. We obtain that F^l is a cylinder with $(l-1)$-dimensional generatrices.

If f', f'' are collinear vectors for some interval u^1, then the curve with the radius vector $f(u^1)$ is a straight line, F^l is a cylinder with one-dimensional generatrices, the directrix F^{l-1} is a hypersurface in E^l isometric to the Euclidean space.

2) As in the proof of Theorem 2.4, consider the mapping

$$\begin{aligned} \tilde{r} &= r - u^1 r_1, \\ \tilde{r}_1 &= -b_{11}\xi, \\ \tilde{r}_j &= 0, \qquad j = 2, \dots, l. \end{aligned}$$

a) Suppose that for some interval u^1, $b_{11} = 0$. Then

$$r - u^1 r_1 = 0.$$

Solving this equation with respect to the vector function r, we get

$$r = u^1\rho(u^2, \dots, u^l),$$

where $\langle \rho, \rho\rangle = 1$. The submanifold F^{l-1} with the radius vector $\rho = \rho(u^2, \dots, u^l)$ is a local isometric embedding of the unit sphere to the unit sphere S^l. If $u^1 \in [0, u_0^1]$, then F^l is the Euclidean space E^l.

b) Suppose that for some interval u^1, $b_{11} \neq 0$. At this case,

$$r - u^1 r_1 = -b_{11}(u^1)\xi(u^1) = \Phi(u^1),$$

where ξ is the unit normal to the hypersurface F^l. By the above,

$$\xi_j = 0, \quad j = 2, \dots l. \tag{3.2}$$

From the Weingarten formulas and (3.2), is follows that

$$b_{j\alpha} = 0, \quad j = 2, \dots, l, \ \alpha = 1, \dots, l. \tag{3.3}$$

Recall the notion of the extrinsic null-index.

Definition 3.2. The extrinsic null-index $\mu(Q)$ of a point Q of submanifold F^l in the Euclidean space E^{l+p} is the maximal dimension of a subspace $L(Q)$ of the tangent space $T_Q F^l$ such that

$$B_\xi y = 0 \tag{3.4}$$

for any vector $y \in L(Q)$ and any normal $\xi \in N_Q F^l$ at this point, where B_ξ is the linear transformation in $T_Q F^l$ corresponding to the second fundamental form with respect to the normal ξ [3].

From (3.3), (3.4), we obtain that in our case the extrinsic null-index $\mu(Q) = l - 1$ for any point $Q \in F^l$ and the subspaces $L(Q)$ are orthogonal to the coordinate lines u^1. The null-index is constant. Then the null-distribution is integrable and the leaves are $(l-1)$-dimentional planes in E^{l+1} and the normals to F^l are stationary along the leaf [3, Lemma 3.1.1]. But at same time, the leaves are to be orthogonal to the coordinate lines u^1, which is impossible because the coordinate lines u^1 are orthogonal to the intrinsic spheres $u^1 = \text{const}$ and case b) is also impossible. □

4. Submanifolds with positive sectional curvature in Euclidean space

Theorem 4.1. *Suppose that F^l is a regular hypersurface with induced metric of revolution of positive sectional curvature in the Euclidean space E^{l+1}.*

1) *If $l \geq 3$, then F^l is a hypersurface of revolution.*

2) *If $l = 2$ and the coordinate lines u^1 are the lines of curvature, then the surface F^2 is a surface of revolution in E^3.*

Proof. 1) A metric of revolution of positive sectional curvature has only one form:

$$ds^2 = (du^1)^2 + \varphi^2(u^1)d\sigma^2,$$

where $d\sigma^2$ is the metric of a unit sphere. The function φ satisfies the following conditions:

$$\varphi(0) = 0, \quad \varphi'(0) = 1, \quad \varphi''(u^1) < 0, \quad \varphi'(u^1) < 1, \quad u^1 > 0.$$

Let $\psi(u^1)$ denote $\int_0^{u^1} \sqrt{1-(\varphi'(t))^2}dt$ and $\rho = \rho(u^2,\ldots,u^l)$ be a radius vector of the unit sphere S^{l-1} in the Euclidean space E^l. Then the hypersurface in E^{l+1} with the radius vector

$$\begin{cases} x^i = \varphi(u^1)\rho^1(u^2,\ldots,u^l); \\ \quad\ldots \\ x^l = \varphi(u^1)\rho^l(u^2,\ldots,u^l); \\ x^{l+1} = \psi(u^1) \end{cases}$$

is a hypersurface of revolution with induced metric of revolution of positive sectional curvature. The rank of the second fundamental form F^l is l. If $l \geq 3$ and the ranks of the second fundamental forms of two isometric hypersurfaces F_1^l, F_2^l in E^{l+1} are greater or equal to 3, then the hypersurfaces coincide up to a rigid motion of the Euclidean space E^{l+1} [4, Theorem 6.2].

2) For $l = 2$, the proof is similar to that of Theorem 2.4. □

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Про структуру багатовимірних підмноговидів з метрикою обертання в евклідовому просторі

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Знайдено умови на зовнішні властивості підмноговидів малої коримірності за яких підмноговид з індукованою метрикою обертання складеної кривини сталого знаку є підмноговидом обертання.

Ключові слова: метрика обертання, підмноговид обертання, лінії кривини, секційна кривина.