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An Estimation of the Length of a Convex Curve in Two-Dimensional Aleksandrov Spaces

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In the paper, a generalization of the Toponogov theorem about the length of a curve in a two-dimensional Riemannian manifold is proved for the case of two-dimensional Aleksandrov spaces.

Key words: λ-convex curve, two-dimensional Aleksandrov space.

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Let R be an Aleksandrov space of curvature $\geq c$ homeomorphic to a disc (see [1, p. 308]). Suppose G is a domain in R that is bounded by a rectifiable curve γ. Denote by $\tau(\gamma_1)$ the integral geodesic curvature (the swerve) of a subarc γ_1 of γ [1, p. 309]. The curve γ is called *λ-convex* with $\lambda > 0$ if any subarc γ_1 of γ satisfies

$$\frac{\tau(\gamma_1)}{s(\gamma_1)} \geq \lambda > 0, \tag{1}$$

where $s(\gamma_1)$ is the length of γ_1. For regular curves in a two-dimensional Riemannian manifold this condition is equivalent to the assumption that the geodesic curvature at each point of this curve is $\geq \lambda > 0$. In the general case the condition (1) allows γ to have corner points.

We prove the following theorem.

Theorem 1. *Let G be a domain homeomorphic to a disc and G lie in a two-dimensional Aleksandrov space of curvature $\geq c$ (in the sense of Aleksandrov).*

I. *If the boundary curve γ of G is λ-convex and $c + \lambda^2 > 0$, then the length $s(\gamma)$ of γ satisfies*

1. $s(\gamma) \leq \dfrac{2\pi}{\lambda}$ *for $c = 0$;*
2. $s(\gamma) \leq \dfrac{2\pi\sqrt{c}}{\sqrt{c+\lambda^2}}$ *for $c > 0$;*
3. $s(\gamma) \leq \dfrac{2\pi\sqrt{-c}}{\sqrt{c+\lambda^2}}$ *for $c < 0$.*

II. *All these inequalities become equalities if and only if the domain G is a disc on the plane of constant curvature c.*

This theorem is a generalization of the Toponogov theorem [10] about the length of a curve in a two-dimensional Riemannian manifold. We need the following statements to prove Theorem 1.

Theorem A (A.D. Aleksandrov [1, p. 269])**.** *A metric space with intrinsic metric of curvature $\geq c$ homeomorphic to a sphere is isometric to a closed convex surface in a simply connected space of constant curvature c.*

Theorem B (A.V. Pogorelov [9, pp. 119–167, 267, 320–321], [4])**.** *Closed isometric convex surfaces in the three-dimensional Euclidean and spherical spaces are equal up to a rigid motion.*

Theorem C (A.D. Milka [8])**.** *Closed isometric convex surfaces in the three-dimensional Lobachevsky space are equal up to a rigid motion.*

Theorem D (W. Blaschke [2])**.** *Let γ be a closed embedded C^2 regular curve in the Euclidean plane.*

I. *If the curvature k of γ at each its point P satisfies*

$$k \geq \lambda > 0,$$

then the curve belongs to the disc that is bounded by the circle of radius $R = 1/\lambda$ tangent to the curve at P.

II. *If the curvature k of γ at each its point P satisfies*

$$0 \leq k \leq \lambda,$$

then the circle of radius $R = 1/\lambda$ tangent to the curve at P belongs to the domain G that is bounded by the curve γ.

Theorem D remains true if the condition for the curvature k of γ is replaced by the same condition for the specific curvature $\frac{\tau(\gamma_1)}{s(\gamma_1)}$ for any arc.

Lemma 1. *Let γ be a closed embedded rectifiable curve in the Euclidean plane.*

I. *If for any subarc γ_1 of γ the specific curvature $\frac{\tau(\gamma_1)}{s(\gamma_1)}$ satisfies*

$$\frac{\tau(\gamma_1)}{s(\gamma_1)} \geq \lambda > 0,$$

then the curve γ belongs to the disc that is bounded by the circle of radius $R = 1/\lambda$ tangent to the support straight line of γ at its point P.

II. *If for any subarc γ_1 of γ the specific curvature $\frac{\tau(\gamma_1)}{s(\gamma_1)}$ satisfies*

$$0 \le \frac{\tau(\gamma_1)}{s(\gamma_1)} \le \lambda,$$

then the circle of radius $R = 1/\lambda$ tangent to the curve at its point P belongs to the domain G that is bounded by the curve γ.

Proof. I. In this case the support function $h(\phi)$, $0 \le \phi \le 2\pi$, of the curve γ is $C^{1,1}$ regular and a.e. it satisfies the equation

$$h + h'' = R, \quad 0 \le R \le \frac{1}{\lambda},$$

where R is the radius of curvature of γ. Therefore,

$$h(\phi) = \int_0^{\phi} R(\sigma)\sin(\phi - \sigma)\, d\sigma,$$

and the proof coincides with Blaschke's proof [2].

II. The position vector $r(s)$ of the curve γ is a $C^{1,1}$ regular vector function. Fix an initial point P_0 on γ and denote by e_1 the unit tangent vector of γ at P_0, and by e_2 the unit normal vector of γ at P_0. Let $P(s)$ be a point on γ such that the length of the arc $\gamma(s) = P_0P(s)$ equals s. The function $\tau(s) = \tau(\gamma(s))$ is the integral geodesic curvature of the arc $\gamma(s)$, and $\tau(s) \le \lambda s$. Therefore,

$$r'(s) = \cos\tau(s)\, e_1 + \sin\tau(s)\, e_2. \tag{2}$$

If we compare (2) with the equation for the circle of radius $1/\lambda$, we obtain the proof. □

H. Karcher proved a generalization of the Blaschke theorem for regular curves in the spherical space $\mathbb{S}^2$ and in the Lobachevsky space $\mathbb{H}^2$ [7]. We formulate Lemma 2 for the cases where the curvature of $\mathbb{S}^2$ is equal to 1 and the curvature of $\mathbb{H}^2$ is equal to -1. Lemma 2 remains true for the planes of any constant curvature c and the proof is the same.

Lemma 2. *Let γ be a closed embedded rectifiable curve in $\mathbb{H}^2$ or $\mathbb{S}^2$.*

I. *If the specific curvature satisfies*

$$\frac{\tau(\gamma_1)}{s(\gamma_1)} \ge \coth R_0 = \lambda$$

for any subarc γ_1 of γ in $\mathbb{H}^2$, then the curve γ belongs to the disc that is bounded by the circle of radius R_0 tangent to the support straight line of γ at a point $P \in \gamma$.

II. *If the specific curvature satisfies*

$$\frac{\tau(\gamma_1)}{s(\gamma_1)} \geq \cot R_0 = \lambda,$$

for any subarc γ_1 of γ in $\mathbb{S}^2$, then the curve γ belongs to the disc that is bounded by the circle of radius R_0 tangent to the support straight line of γ at a point $P \in \gamma$.

Proof. The curve γ is a closed convex curve. At any point P of γ there exists a support straight line (a geodesic line in the plane of constant curvature).

I: $\gamma \in \mathbb{H}^2$. Let S be a circle of radius R_0 tangent to the support straight line of γ from the side containing γ. Assume that the center of the circle S is the origin of the coordinate system in the Cayley–Klein model of the Lobachevsky plane and it is also the origin for the support function h of the curve γ. The support function h is $C^{1,1}$ regular and a.e. the radius of curvature R of γ equals

$$R = \frac{g + g''}{\left(1 - \frac{(g')^2}{1+g^2}\right)^{3/2}}, \tag{3}$$

where $g(h) = \tanh h$ is the support function for the curve $\widetilde{\gamma}$, and $\widetilde{\gamma}$ is the image of γ under the geodesic map from $\mathbb{H}^2$ to $\mathbb{E}^2$ [5, 6]. The radius of curvature $\widetilde{R}$ of $\widetilde{\gamma}$ is a.e. equal to

$$\widetilde{R} = R\left(1 - \frac{(g')^2}{1+g^2}\right)^{3/2}, \quad 0 \leq \widetilde{R} \leq R. \tag{4}$$

The image of the circle S under the geodesic map is the circle $\widetilde{S}$ in the Euclidean plane $\mathbb{E}^2$ with the center at the origin of a Cartesian orthogonal coordinate system. The curvature of $\widetilde{S}$ equals $\coth R_0$. From Lemma 1(I), it follows that $\widetilde{\gamma}$ belongs to the disc bounded by the circle $\widetilde{S}$. Applying the inverse geodesic transformation, we obtain that the curve γ belongs to the disc that is bounded by the circle S in the Lobachevsky plane $\mathbb{H}^2$.

II: $\gamma \in \mathbb{S}^2$. Let $\overline{\gamma}$ be the polar to the curve γ in $\mathbb{S}^2$. The position vector of $\overline{\gamma}$ is $C^{1,1}$ regular and its curvature is $\leq \tan R_0$ a.e. Let P_0 be a point on γ and $\overline{S}$ be a circle of radius $\pi/2 - R_0$ tangent to $\overline{\gamma}$ at the point $\overline{P}_0$. The curvature of this circle is equal to $\tan R_0$. The center $\overline{O}$ of $\overline{S}$ is the south pole of the sphere. Consider the geodesic map of the sphere $\mathbb{S}^2$ into the plane tangent to $\mathbb{S}^2$ at the point $\overline{O}$. The curve $\overline{\gamma}$ is mapped to a curve $\widetilde{\overline{\gamma}} \in \mathbb{E}^2$, and the circle $\overline{S}$ is mapped to a circle $\widetilde{\overline{S}}$ of curvature $\tan R_0$. The curvatures satisfy $\widetilde{\overline{k}}(\widetilde{\overline{\gamma}}) \leq \overline{k}(\overline{\gamma}) \leq \tan R_0$. From Lemma 1(II) it follows that the circle $\widetilde{\overline{S}}$ belongs to the domain that is bounded by the curve $\widetilde{\overline{\gamma}}$. Applying the inverse geodesic transformation, we obtain that the circle $\overline{S}$ belongs to the domain bounded by $\overline{\gamma}$ and the polar curve γ belongs to the disc bounded by the polar circle S of radius R_0. □

Proof of Theorem 1. Let G_1 and G_2 be two copies of the domain G. Let us glue the domains G_1 and G_2 along their boundary curves γ_1 and γ_2 by an isometry between these curves. We obtain a manifold F homeomorphic to the two-dimensional sphere with an intrinsic metric. Since the sum of the integral geodesic curvatures of any two identified arcs of the boundary curves is non-negative, from the Aleksandrov gluing theorem [1, p. 318] it follows that F is an Aleksandrov space of curvature $\geq c$. By Theorem A, this manifold can be isometrically embedded as a closed convex surface F_1 in the simply-connected space $M^3(c)$ of constant curvature c. From Theorem B and C it follows that this surface is unique up to a rigid motion.

By *plane domains* we will understand domains on totally geodesic two-dimensional surfaces in spaces of constant curvature; similarly we will call geodesic lines in these spaces as *lines*.

Perform the reflection of the surface F_1 with respect to a plane π passing through three points on γ that do not belong to any line. We will get a mirrored surface F_2. The domains G_1 and G_2 are mapped to domains $\widetilde{G}_1$ and $\widetilde{G}_2$ on F_2; the curve γ is mapped to $\widetilde{\gamma}$. But G_1 is isometric to G_2 and $\widetilde{G}_2$ is isometric to $\widetilde{G}_1$. Let us reverse the orientation of the domains $\widetilde{G}_1$, $\widetilde{G}_2$. Then the surface F_2 will be isometric to F_1 and they will have the same orientation. By Theorems B and C, the surface F_1 can be mapped to the surface F_2 by a rigid motion of the ambient space. But the three points of the curve γ are fixed under this rigid motion. Thus it follows that this motion is the identity mapping and, moreover, the curve γ coincides with the curve $\widetilde{\gamma}$. Such a situation is possible only when the curve γ is a plane curve and it is the boundary of a convex cup isometric to the domain G. Recall that the *convex cup* is a convex surface with a planar boundary curve γ such that the surface is a graph over a plane domain $\overline{G}$ enclosed by γ. Note that since γ is a convex curve on the plane, the integral geodesic curvature of any arc of the curve γ is non-negative if γ is viewed as a curve on the cup and as a curve on a plane [3].

Let us show that the integral geodesic curvature of any arc of γ calculated on G is not less than its corresponding integral geodesic curvature calculated on the cup G. This means that γ as the boundary curve of $\overline{G}$ is also λ-convex.

Recall that the intrinsic curvature $\omega(D)$ of a Borel set D on a convex surface in a space of constant curvature c is

$$\omega(D) = \psi(D) + cF(D),$$

where $\psi(D)$ is the extrinsic curvature, $F(D)$ is the area of D [1, p. 397]. Consider a closed convex surface M bounded by G and the plane domain $\overline{G}$, and a surface $\overline{M}$ composed of the double-covered domain $\overline{G}$.

The intrinsic curvature concentrated on γ equals

$$\omega(\gamma) = \tau_\gamma(G) + \tau_\gamma(\overline{G}),$$

where $\tau_\gamma(G)$, $\tau_\gamma(\overline{G})$ are the integral geodesic curvatures of γ computed in G and $\overline{G}$ respectively.

Since $F(\gamma) = 0$, we have

$$\psi_M(\gamma) = \tau_\gamma(G) + \tau_\gamma(\overline{G}),$$

$$\psi_{\overline{M}}(\gamma) = 2\tau_\gamma(\overline{G}).$$

From the definition of the extrinsic curvature [1, p. 398] it follows that $\psi_{\overline{M}}(\gamma) \geq \psi_M(\gamma)$ because each plane supporting M at a point of γ is also supporting $\overline{M}$. Thus we obtain $\tau_\gamma(\overline{G}) \geq \tau_\gamma(G)$. Moreover, this inequality holds for any subarc of γ.

I. The curve γ is a λ-convex curve lying in the plane of constant curvature c. From Lemmas 1 and 2, it follows that the curve γ belongs to a disc bounded by the circle of radius R_0. The curvature and the length s of this circle equal

1. $\lambda = \frac{1}{R_0}$, $\quad s = 2\pi R_0 \quad$ for $c = 0$;
2. $\lambda = \sqrt{c}\cot\sqrt{c}R_0$, $\quad s = 2\pi\sin\sqrt{c}R_0 \quad$ for $c > 0$;
3. $\lambda = \sqrt{-c}\coth\sqrt{-c}R_0$, $\quad s = 2\pi\sinh\sqrt{-c}R_0 \quad$ for $c < 0$.

The curve γ on the plane of constant curvature c bounds the convex domain G. It follows that the length of γ satisfies

$$s(\gamma) \leq \begin{cases} \dfrac{2\pi}{\lambda} & \text{if } c = 0 \\ \dfrac{2\pi\sqrt{c}}{\sqrt{c+\lambda^2}} & \text{if } c > 0 \\ \dfrac{2\pi\sqrt{-c}}{\sqrt{c+\lambda^2}} & \text{if } c < 0. \end{cases} \tag{5}$$

II. Suppose that there is equality in (5). Then the domain $\overline{G}$ is a disc bounded by the circle γ. Furthermore, $\tau_\gamma(\overline{G}) = \tau_\gamma(G)$ and the intrinsic curvature of γ satisfies $\omega_M(\gamma) = \omega_{\overline{M}}(\gamma) = 2\tau_\gamma(\overline{G})$, and the extrinsic curvature for any subarc γ_1 of γ satisfies

$$\psi_M(\gamma) = \psi_{\overline{M}}(\gamma). \tag{6}$$

It follows that the surfaces M and $\overline{M}$ coincide, M is a double-covered disk and thus G is a disk. If M does not coincide with $\overline{M}$, then there exists a set of a positive measure of supporting planes to $\overline{M}$ along γ which are not supporting planes to M. It follows that the extrinsic curvatures of M and $\overline{M}$ along γ do not coincide. This contradicts equality (6), and thus Theorem 1 is proved. □

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Про оцінку довжини опуклої кривої у двовимірному просторі Александрова

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Доведено узагальнення теореми Топоногова про довжину кривої у двовимірному рімановому просторі на випадок двовимірного простору Александрова.

Ключові слова: λ-опуклі криві, двовимірний простір Александрова