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BALANCING PRINCIPLE FOR ITERATED TIKHONOV METHOD OF SEVERELY ILL-POSED PROBLEMS

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Резюме. В даній статті розглядається проблема наближеного розв'язування жорстко некоректних задач зі збуреними вхідними даними. До регулювання таких задач було застосовано ітерований метод Тіхонова з правилом зупинки згідно принципу рівноваги. Для запропонованого підходу була знайдена порядкова оцінка похибки на класі задач, що досліджуються.

Abstract. Considered in this paper are the problem of approximate solving severely ill-posed problems with perturbed input data. In oder to regularize these problems the iterated Tikhonov method with balancing principle as stop rule was applied. For this suggesting approach an order of accuracy on the class of problems under investigation was found.

1. Introduction

In this paper we consider the problem of approximate solving severely ill-posed problems represented in the form of operator equation of the first kind

$$Ax = y, \tag{1}$$

where $A : X \to Y$ is linear compact injective operator between Hilbert spaces X and Y. Let us denote inner products in these spaces by $(\cdot, \cdot)$ and corresponding norms by $\|\cdot\|$. The symbol $\|\cdot\|$ stands also for standart operator norm. It will become clear from the context which exactly space or norm is under consideration. Suppose also that an available perturbation $y_\delta \in Y : \|y - y_\delta\| \leq \delta, \delta > 0$, is known instead of the right-hand side y and a perturbed operator $A_h : \|A - A_h\| \leq h, h > 0$, is known instead of A, where $A_h : X \to Y$ is also linear compact injective one.

Usually, equation (1) is referred to as a severely ill-posed problem if its solution $x_0 = A^{-1}y$ has a finite "smoothness" in some sense, but A is an infinitely smoothing operator.

A distinguishing characteristic of such kind of problems is the fact that x_0 belongs to some subspace V continuously embedded in X, the singular values of the canonical embedding operator J_V from V into X tend to zero with polynomical rate, while the singular values $\{\sigma_l\}_{l=1}^{\infty}$ of the operator A tend to zero exponentially.

Following [2], [7] suppose that x_0 belongs to the set

$$M_{p,\rho}^{K}(A) := \{x : x = (\underbrace{\ln \ldots \ln}_{K\text{-times}}(A^*A)^{-1})^{-p}v, \quad \|v\| \leq \rho\}, \tag{2}$$

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†*Key words.* Severely ill-posed problem, balancing principle, iterated Tikhonov method.

when some unknown $0 < p \le p_1$, $K = 1,2,...$, and known $\rho > 0$, where the operator function $(\underbrace{\ln ... \ln}_{K\text{-times}}(A^*A)^{-1})^{-p}$ well defined by the spectral decomposition

$$A^*A = \sum_{l=1}^{\infty} \sigma_l^2 (\Psi_l, \cdot)\Psi_l$$

of the operator A^*A, i.e.

$$(\underbrace{\ln ... \ln}_{K\text{-times}}(A^*A)^{-1})^{-p} v = \sum_{l=1}^{\infty} (\underbrace{\ln ... \ln}_{K\text{-times}}(\sigma_l^{-2}))^{-p} (\Psi_l, v)\Psi_l.$$

Further, without loss of generality we assume that

$$\|A\| \le M_K,\ M_K = m_K^{1/2}, m_k = \begin{cases} e^{-1}, & k = 1, \\ e^{-\frac{1}{m_{k-1}}}, & k = 2, ..., K \end{cases},$$

i.e.

$$\sigma_l \le m_K, \quad l = 1, 2, ... \ .$$

Example 1. To illustrate severely ill-posed problems let us consider a problem from satellite gravity gradiometry. With the surfaces of the Earth and the satellite orbit assumed to be sphericals with radius $r_1 < r_2$, correspondently, $\Omega_{r_i} = \left\{ u \in \mathbb{R}^3, |u| = r_i \right\}, i = 1,2$, then one of the problems arising in this theory (see, e.g.,[4], [11]) could be formulated as an equation (1) with the operator

$$Ax(u) := \frac{1}{4\pi r_1} \int\limits_{\Omega_{r_1}} \frac{d^2}{dr_2^2} \left(\frac{r_2^2 - r_1^2}{|u - v|^3} \right) x(v) d\Omega_{r_1}(v), \quad u \in \Omega_{r_2}. \qquad (3)$$

In satellite gradiometry the exact solution of equation (1) with operator (3) is usually considered to be an element of the spherical Sobolev space

$$\mathcal{H}^s := \{ f \in L_2(\Omega_{r_1}) : \|f\|_s^2 = \sum_{l=0}^{\infty} \sum_{k=1}^{2l+1} \left(l + \frac{1}{2} \right)^{2s} |\langle Y_{l,k}^{(1)}, f \rangle|^2 < \infty \ \}$$

for some positive index s, where

$$Y_{l,k}^{(1)}(\omega) = \frac{1}{r_1} Y_{m,j}(\frac{\omega}{r_1}), \quad \omega \in \Omega_{r_1},$$

$$\langle Y_{l,k}^{(1)}, x \rangle = \int\limits_{\Omega_{r_1}} Y_{l,k}^{(1)}(v) x(v) d\Omega_{r_1}(v)$$

and $\{Y_{m,j}, m = 0,1,...,j = 1,2,...,2m+1\}$ is a set of spherical harmonics L_2-orthonormalized with respect to the unit sphere in $\mathbb{R}^3$.
As for the singular values σ_l of the operator (3) the following relation (see, e.g., [12])

$$\ln \sigma_l^{-2} \asymp l + \frac{1}{2}$$

is valid, then there are some constants $c_2 > c_1 > 0$ such that for any $f \in \mathcal{H}^s$ two-sided estimate

$$c_1 \|f\|_s \leq \| \ln^s (A^*A)^{-1} f\| \leq c_2 \|f\|_s$$

is valid. It, in particular, means that any element of $\mathcal{H}^s$ belongs to the set (2) with $K = 1$ and $p = s$.

Example 2. Let us consider a two-dimensional model of the scattering by a perfectly reflecting periodic structure. According to Bao [3], Hettlich and Kirsch [5], we can formulate the problem as follows. Let $f \in C^2(\mathbb{R})$ be 2π-periodic function with $f(x) > 0$ for all $x \in \mathbb{R}$. We set

$$\Omega_f = \{(x, y) : y > f(x), x \in \mathbb{R}\}.$$

Then by

$$\partial\Omega_f = \{(x, y) : y = f(x), x \in \mathbb{R}\}$$

we denote a periodic interface which should be determined from scattering data. For this end, we introduce an incident field $u^I(x, y; k)$ given by

$$u^I(x, y; k) = \exp\{ik(x \sin\theta - y \cos\theta)\}, \tag{4}$$

which is a time-harmonic electromagnetic plane wave. Here $i = \sqrt{-1}$ and the constant $k \in \mathbb{R}$ is the refraction index of the material occupying Ω_f, and is given by $k = \omega c_0^{-1}\sqrt{\varepsilon\mu}$, where ω is the angular frequency, c_0 is the speed of light, $\mu > 0$ is the magnetic permeability and ε is the dielectric coefficient. Moreover, in (4), θ is regarded as the angle of incidence.

We assume that

$$0 < |\theta| < \frac{\pi}{2}$$

and

$$0 < k < \frac{1}{2\pi}.$$

Then the resulting scattering field $u^S(x, y; k)$ satisfies the Helmholtz equation with the perfect reflection boundary condition

$$\Delta u^S + k^2 u^S = 0 \text{ in } \Omega_f, \tag{5}$$

$$u^S + u^I = 0 \text{ on } \partial\Omega_f, \tag{6}$$

u^S satisfies so-called outgoing wave condition:

$$u^S = \sum_{n\in\mathbb{Z}} u_n e^{i(\alpha_n x + \beta_n y)}, \text{ if } y > \|f\|_{C[0;2\pi]}. \tag{7}$$

In this example the function u^S under consideration is regarded as complex-valued. Here, we set

$$\alpha_n = n + k\sin\theta, \quad \beta_n = \sqrt{k^2 - (n + k\sin\theta)^2}, \quad 0 \leq \arg\beta_n < \pi. \tag{8}$$

Moreover, we impose the $(k\sin\theta)$-quasi-periodicity condition over u^S

$$u^S(x + 2\pi, y; k) = \exp(2\pi i k \sin\theta) u^S(x, y; k) \tag{9}$$

for all $(x, y) \in \mathbb{R}^2$ (see, e.g., [3]).

Now we can state our inverse problem.

Determine $y = f(x), \quad x \in \mathbb{R}$, from measurement $u^S(x, y; k)$, $x \in (0; 2\pi)$, where u^S satisfies (5)-(7) and (9).

By the $(k \sin\theta)$-quasi-periodicity, setting

$$u = u(x, y; k) = u^I(x, y; k) + u^S(x, y; k).$$

We can rewrite (5)-(7) and (9) in terms of the total field u:

$$\Delta u + k^2 u = 0 \text{ in } \Omega_f, \tag{10}$$

$$u = 0 \text{ on } \partial\Omega_f, \tag{11}$$

$$u(x + 2\pi, y; k) = \exp(2\pi i k \sin\theta) u(x, y; k), \tag{12}$$

$$u - u^I \text{ satisfies the outgoing wave condition.} \tag{13}$$

Since k is fixed such that (8) is true, we simply write $u(x, y)$ in place of $u(x, y; k)$. Then our inverse problem is equivalent to determine $y = f(x), \quad x \in \mathbb{R}$, from measurement

$$u(x, 0), x \in (0; 2\pi),$$

where u satisfies (10)-(13).

For fixed positive constants $M_0, \quad M, \quad k$ and $a_0, \quad a$ such that $0 < M \le a_0 \le a$ and $0 < k < 1$, we set

$$\mathcal{F} = \{f \in C^{3+k}(\mathbb{R}) : \|f\|_{C^{3+k}[0;2\pi]} \le M_0, \quad f \quad \text{is} \quad 2\pi\text{-periodic},$$
$$\frac{d^j f}{dx^j}(0) = \frac{d^j f}{dx^j}(2\pi), \quad j = 0, 1, 2, 3,$$
$$f(0) = f(2\pi) = -a_0, \quad -a \le f(x) \le -M,$$
$$0 \le x \le 2\pi\}$$

as an admissible set of unknown surfaces.

Denote

$$\|f\|_{C^{3+k}[0;2\pi]} = \sum_{j=0}^{3} \left\| \frac{d^j f}{dx^j} \right\|_{C[0;2\pi]} + \sup_{0 < x, x' \le 2\pi, x \ne x'} \frac{|(\frac{d^3 f}{dx^3})(x) - (\frac{d^3 f}{dx^3})(x')|}{|x - x'|^k}.$$

Let us set

$$\Omega_f = \{(x, y) : y > f(x), x \in \mathbb{R}\} \text{ for } f \in \mathcal{F}.$$

For $f_j \in \mathcal{F}, j = 1, 2$, let us consider

$$\Delta u_j + k^2 u_j = 0 \text{ in } \Omega_{f_j},$$
$$u_j = 0 \text{ on } \partial\Omega_{f_j},$$
$$u_j \text{ is } (k \sin\theta)\text{-quasi-periodicity, i.e.}$$
$$u_j(x + 2\pi, y) = \exp(2\pi i k \sin\theta) u_j(x, y).$$

We further assume that $u_j - u^I$ satisfies the outgoing wave condition.

Theorem (2.1) [5] shows that in stated above conditions there exists a constant $C = C(k, \theta, \mathcal{F}) > 0$ such that

$$\|f_1 - f_2\|_{C[0;2\pi]} \le \frac{C}{|\ln|\ln \frac{1}{\|(u_1 - u_2)(\cdot, 0)\|_{H^1(0;2\pi)}}||}$$

provided that for all $f_1, f_2 \in \mathcal{F}$. Hence, solution of equation (5) belongs to the set (2) with $K = 2$ and $p = 1$.

As far as the history of studing severely ill-posed problems, we should notice, that these studies could be traced back to work [8], where the estimate of accuracy for the Tikhonov regularization were found for equations (1) with operators of both finite and infinite smoothness. Moreover, some regularization methods for severely ill-posed problems were considered in [6], where, in particular, a general class regularization methods (according to Bakushinskiy; see, e.g., [1]) were suggested for solving (1) in the case of perturbed operators and the right-hand sides; for choosing a regularization parameter was employed a modification from [10]. Further, severely ill-posed problems were considered, in particular, in works [7], [2], [12], [13]. In [12] the approch for solving ill-posed problems (1) with solutions from (2) for $K = 1$ was proposed. It suggests a combination of usual Tikhonov's regularization with Morozov's discrepancy principle. The indicated combination allows to achieve the order-optimal accuracy (in the logarithmic scale) $O(\ln^{-1}\frac{1}{\delta})$ of recovering solution from the set $M^1_{p,\rho}(A)$ for any $p > p_0 > 0$. In [13] for solving the same problem Tikhonov's method was employed again; however, for the stop rule was considered the balancing principle. This approach also allows to attain the order-optimal accuracy $O(\ln^{-1}\frac{1}{\delta})$ of recovering solutions from pointed set for all $0 < p \le 1$. Notice, that studies initiated in [12] were extended in [14] to the more wide class of ill-posed problems (1) with solutions (2) for any $K = 1, 2, ...$ and $p > p_0 > 0$. Herewith the order-optimal accuracy of recovering solutions $O((\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\delta})^{-p})$ was obtained.

Unlike the works described above, in the present paper for regularization of severely ill-posed problems (1) with solutions (2) for $K \ge 1$, and perturbed operators and the right-hand sides iterated Tikhonov's method will be applied, and a regularization parameter will be chosen in accordance with the balancing principle. Subsequently we will demonstrate that the suggested approach for solving (1)-(2), which consists in combination of iterative Tikhonov's method and balancing principle, provides accuracy $O((\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{h+\delta})^{-p})$.

We recall that iterated Tikhonov's method consists in a choosing a natural m, initial approximation $x^{h,\delta}_{0,\alpha}$, and consistently computation of elements $x^{h,\delta}_{i,\alpha}$, $i = 1, 2, ..., m$, by the rule

$$x^{h,\delta}_{i,\alpha} = \alpha(A_h^*A_h + \alpha I)^{-1}x^{h,\delta}_{i-1,\alpha} + \alpha(A_h^*A_h + \alpha I)^{-1}A_h^*y_\delta, \tag{14}$$

where $m \ge p_1$ and as the approximate solution we take $x^{h,\delta}_{m,\alpha}$. If $x^{h,\delta}_{0,\alpha} = 0$ then the element $x^{h,\delta}_{m,\alpha}$ can be rewritten in the form of

$$x^{h,\delta}_{m,\alpha} = \sum_{i=1}^{m} \alpha^{i-1}(A_h^*A_h + \alpha I)^{-i}A_h^*y_\delta. \tag{15}$$

Obviously, any numerical realization of the Tikhonov method requires us to curry out all computations with a finite-dimensional approximation $A_{h,n}$ instead of A_h. Thus we assume finite-dimensional approximation $A_{h,n}$ with $\text{rank}(A_{h,n}) = n$ to be chosen such that

$$\|A_h - A_{h,n}\| \le \varepsilon, \text{ where } \varepsilon = \begin{cases} \delta\rho^{-1} & , \quad 0 < h \le \delta, \\ h & , \quad h > \delta \end{cases}. \tag{16}$$

Further, along with (15) we will also consider auxiliary elements:

$$x_{m,\alpha} = \sum_{i=1}^{m} \alpha^{i-1}(A^*A + \alpha I)^{-i} A_h^* y, \tag{17}$$

$$x_{m,\alpha,n}^h = \sum_{i=1}^{m} \alpha^{i-1}(A_{h,n}^*A_{h,n} + \alpha I)^{-i} A_{h,n}^* y, \tag{18}$$

$$x_{m,\alpha,n}^{h,\delta} = \sum_{i=1}^{m} \alpha^{i-1}(A_{h,n}^*A_{h,n} + \alpha I)^{-i} A_{h,n}^* y_\delta. \tag{19}$$

Recall that generating function of the iterated Tikhonov method has the form (see [15, p.21])

$$g_{m,\alpha}(\lambda) := \sum_{i=1}^{m} \alpha^{i-1}(\alpha + \lambda)^{-i} = \frac{1}{\lambda}(1 - \frac{\alpha^m}{(\alpha+\lambda)^m}), \quad \lambda \ne 0,$$

and satisfies inequality (see [15, p.22])

$$\sup_{0<\lambda<\infty} \sqrt{\lambda} g_{m,\alpha}(\lambda) \le \sqrt{\frac{m}{\alpha}}.$$

2. Auxiliary statements

We shall later need the following auxiliary results and facts.

Thus, for any linear operators $A, B \in \mathcal{L}(X, Y)$ and natural m the decomposition (see [15, p. 92])

$$A^m - B^m = \sum_{j=0}^{m-1} A^j (A - B) B^{m-j-1} \tag{20}$$

holds true.

Lemma 1. *(see* [15, p. 34]*) If g is bounded, Borel measurable function with respect to the* $[0; M_K]$,
$A \in \mathcal{L}(X, Y)$, $\|A\| \le M_K$ *then*

$$A^* g(AA^*) = g(A^*A)A^*,$$

$$Ag(A^*A) = g(AA^*)A.$$

In addition, it is well-known that for any bounded linear operator B

$$B(\alpha I + B^*B)^{-1} = (\alpha I + BB^*)^{-1}B,$$
$$\|(\alpha I + B^*B)^{-1}\| \le \alpha^{-1}, \quad \|(\alpha I + B^*B)^{-1}B^*\| \le \frac{1}{2\sqrt{\alpha}}, \tag{21}$$
$$\|B(\alpha I + B^*B)^{-1}B^*\| \le 1$$

hold.

Before proceeding further we establish a nomber of auxiliary assortations which will be need later for analysis of approximating properties of suggesting approch.

Lemma 2. *Let*

$$\|A\| \le M_K,\ M_K = m_K^{1/2},\ m_k = \begin{cases} e^{-1}, & k=1, \\ e^{-\frac{1}{m_{k-1}}}, & k=2,...,K \end{cases}.$$

Then the following estimate

$$\|x_0 - x_{m,\alpha}\| \le \rho(\underbrace{\ln ... \ln}_{K\text{-}times}\frac{1}{\alpha})^{-p}$$

holds true, where $x_{m,\alpha}$ determined by (17).

Proof. First, we note that

$$\|x_0 - x_{m,\alpha}\| = \|[(\underbrace{\ln ... \ln}_{K\text{-times}}(A^*A)^{-1})^{-p}v -$$
$$\sum_{i=1}^{m}\alpha^{i-1}(A^*A+\alpha I)^{-i}A^*A(\underbrace{\ln ... \ln}_{K\text{-times}}(A^*A)^{-1})^{-p}v]\| \le$$
$$\le \rho\|[I - \sum_{i=1}^{m}\alpha^{i-1}(A^*A+\alpha I)^{-i}A^*A](\underbrace{\ln ... \ln}_{K\text{-times}}(A^*A)^{-1})^{-p}\| \le$$
$$\le \rho \sup_{0<\lambda\le m_K}|[I - \sum_{i=1}^{m}\alpha^{i-1}\frac{\lambda}{(\lambda+\alpha)^i}](\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda})^{-p}| \le$$
$$\le \rho \sup_{0<\lambda\le m_K}|(\frac{\alpha}{\alpha+\lambda})^m(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda})^{-p}|.$$

To estimate the expression standing under sign of supremum we consider two events:

1) $\lambda \le \alpha$. As function $(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda})^{-p}$ monotonously decreases for λ, then

$$(\frac{\alpha}{\alpha+\lambda})^m(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda})^{-p} < (\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\alpha})^{-p}.$$

2) $\lambda \ge \alpha$. We consider the function

$$f(\lambda) = \frac{1}{\lambda^m}(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda})^{-p}, \quad \lambda \in (0; m_K].$$

It is easy to show that

$$f'(\lambda) = \lambda^{-m-1}(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda})^{-p-1}(\underbrace{\ln ... \ln}_{(K-1)\text{-times}}\frac{1}{\lambda})^{-1}...(\ln\frac{1}{\lambda})^{-1}\times$$
$$\times[p - m\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda}\underbrace{\ln ... \ln}_{(K-1)\text{-times}}\frac{1}{\lambda}\cdot ... \cdot \ln\frac{1}{\lambda}].$$

As $f'(\lambda) < 0$ for $p < m$ consequently $f(\lambda)$ monotonously decreases for $p < m,\ \ m > 0$. Thus,

$$f(\lambda) \le f(\alpha) \text{ for } \lambda \ge \alpha \text{ and}$$
$$\left(\frac{\alpha}{\alpha+\lambda}\right)^m (\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda})^{-p} = \left(\frac{\alpha}{\alpha+\lambda}\right)^m \cdot \lambda^m \cdot \frac{1}{\lambda^m}(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda})^{-p} \le$$
$$\le \frac{\lambda^m}{(\alpha+\lambda)^m}(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\alpha})^{-p} \le (\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\alpha})^{-p}.$$

Herewith, in general case we have

$$\|x_0 - x_{m,\alpha}\| \le \rho(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\alpha})^{-p},$$

hence, the proof is completed. □

Lemma 3. *Let*

$$\|A\| \le M_K,\ M_K = m_K^{1/2},\ m_k = \begin{cases} e^{-1}, & k = 1, \\ e^{-\frac{1}{m_{k-1}}}, & k = 2, ..., K \end{cases}.$$

Then the estimate

$$\|x_{m,\alpha} - x^h_{m,\alpha,n}\| \le \frac{\rho(m+\sqrt{m})(h+\varepsilon)}{\sqrt{\alpha}}$$

holds true, where $x_{m,\alpha}$ *and* $x^h_{m,\alpha,n}$ *determened by (17), (18) correspondently.*

Proof. Clearly, that

$$\|x_0\| = \|(\underbrace{\ln ... \ln}_{K\text{-times}}(A^*A)^{-1})^{-p}v\| \le$$
$$\le \rho \sup_{0<\lambda\le m_K}|(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\lambda})^{-p}| \le \rho.$$
$$\|A - A_{h,n}\| \le \|A - A_h\| + \|A_h - A_{h,n}\| \le h + \varepsilon.$$

Further, we estimate the norm

$$\|x_{m,\alpha} - x^h_{m,\alpha,n}\| = \|g_{m,\alpha}(A^*A)A^*y - g_{m,\alpha}(A^*_{h,n}A_{h,n})A^*_{h,n}y\| =$$
$$= \|g_{m,\alpha}(A^*A)A^*Ax_0 - g_{m,\alpha}(A^*_{h,n}A_{h,n})A^*_{h,n}Ax_0\| \le$$
$$\le \rho\|g_{m,\alpha}(A^*A)A^*A - g_{m,\alpha}(A^*_{h,n}A_{h,n})A^*_{h,n}A\|.$$

We consider the expression standing under norm's sign:

$$g_{m,\alpha}(A^*A)A^*A - g_{m,\alpha}(A^*_{h,n}A_{h,n})A^*_{h,n}A =$$
$$= g_{m,\alpha}(A^*A)A^*A - g_{m,\alpha}(A^*_{h,n}A_{h,n})A^*_{h,n}A_{h,n} +$$

$$+g_{m,\alpha}(A_{h,n}^*A_{h,n})A_{h,n}^*A_{h,n} - g_{m,\alpha}(A_{h,n}^*A_{h,n})A_{h,n}^*A = I_1 + I_2,$$

where

$$I_1 := g_{m,\alpha}(A^*A)A^*A - g_{m,\alpha}(A_{h,n}^*A_{h,n})A_{h,n}^*A_{h,n},$$

$$I_2 := g_{m,\alpha}(A_{h,n}^*A_{h,n})A_{h,n}^*A_{h,n} - g_{m,\alpha}(A_{h,n}^*A_{h,n})A_{h,n}^*A.$$

Now we estimate each of summands I_1, I_2.

Thus,

$$I_1 = (I - \alpha^m(\alpha I + A^*A)^{-m} - (I - \alpha^m(\alpha I + A_{h,n}^*A_{h,n})^{-m}) =$$

$$= \alpha^m[(\alpha I + A_{h,n}^*A_{h,n})^{-m} - (\alpha I + A^*A)^{-m}].$$

We apply the formula (20) to expression standing in braces:

$$I_1 = \alpha^m \sum_{j=0}^{m-1} (\alpha I + A_{h,n}^*A_{h,n})^{-j} \cdot [(\alpha I + A_{h,n}^*A_{h,n})^{-1} - (\alpha I + A^*A)^{-1}]\times$$

$$\times(\alpha I + A^*A)^{-m+j+1} = \alpha^m \sum_{j=0}^{m-1} (\alpha I + A_{h,n}^*A_{h,n})^{-j-1}(A^*A - A_{h,n}^*A_{h,n})\times$$

$$\times(\alpha I + A^*A)^{-m+j} = \alpha^m \sum_{j=0}^{m-1} (\alpha I + A_{h,n}^*A_{h,n})^{-j-1}(A^* - A_{h,n}^*)A\times$$

$$\times(\alpha I + A^*A)^{-m+j} + \alpha^m \sum_{j=0}^{m-1} (\alpha I + A_{h,n}^*A_{h,n})^{-j-1}A_{h,n}^*(A - A_{h,n})\times$$

$$\times(\alpha I + A^*A)^{-m+j}.$$

Whence by Lemma 1 and estimates (21) we obtain

$$\|I_1\| \le \sum_{j=0}^{m-1} [\|(\alpha I + A_{h,n}^*A_{h,n})^{-1}\|^{j-1}\|(\alpha I + AA^*)^{-m+j}A\| +$$

$$+\|(\alpha I + A_{h,n}^*A_{h,n})^{-j-1}A_{h,n}^*\| \cdot \|(\alpha I + A^*A)^{-1}\|^{m-j}]\times$$

$$\times\alpha^m\|A - A_{h,n}\| = \sum_{j=0}^{m-1} [\alpha^{-j-1}\|(\alpha I + A^*A)^{-m+j-1}\|\times$$

$$\times\|(\alpha I + A^*A)^{-1}A\| + \|(\alpha I + A_{h,n}^*A_{h,n})^{-1}\|^j\times$$

$$\times\|(\alpha I + A_{h,n}^*A_{h,n})^{-1}A_{h,n}^*\|\alpha^{-m+j}]\alpha^m\|A - A_{h,n}\| \le$$

$$\le \sum_{j=0}^{m-1} [\alpha^{-j-1} \cdot \alpha^{-m+j+1} \cdot \frac{1}{2\sqrt{\alpha}} + \alpha^{-j}\frac{1}{2\sqrt{\alpha}} \cdot \alpha^{-m+j}]\alpha^m\times$$

$$\times\|A - A_{h,n}\| = \sum_{j=0}^{m-1} \frac{1}{\sqrt{\alpha}}\|A - A_{h,n}\| \le \frac{m}{\sqrt{\alpha}}(h + \varepsilon). \qquad (22)$$

Then, due to (21) we find

$$\|I_2\| = \|g_{m,\alpha}(A_{h,n}^*A_{h,n})A_{h,n}^*(A_{h,n} - A)\| \le$$

$$\leq \sup_{0<\lambda\leq m_K} |\sqrt{\lambda} g_{m,\alpha}(\lambda)| \cdot \|A_{h,n} - A\| \leq \frac{\sqrt{m}}{\sqrt{\alpha}}(h+\varepsilon). \tag{23}$$

Summarizing relations (22) and (23) we finally obtain

$$\|x_{m,\alpha,n} - x^h_{m,\alpha,n}\| \leq \rho \left(\frac{m}{\sqrt{\alpha}}(h+\varepsilon) + \frac{\sqrt{m}}{\sqrt{\alpha}}(h+\varepsilon) \right) =$$

$$\frac{\rho(m+\sqrt{m})(h+\varepsilon)}{\sqrt{\alpha}}.$$

Thus, Lemma is proved. □

Theorem 1. *Let*

$$\|A\| \leq M_K, \;\; M_K = m_K^{1/2}, \;\; m_k = \begin{cases} e^{-1}, & k=1, \\ e^{-\frac{1}{m_{k-1}}}, & k=2,...,K \end{cases}$$

and $x_0 = A^{-1}y \in M^K_{p,\rho}(A)$.

Then the estimate

$$\|x_0 - x^{h,\delta}_{m,\alpha,n}\| \leq \rho((\underbrace{\ln ... \ln}_{K\text{-}times}\frac{1}{\alpha})^{-p} + \frac{\rho(m+\sqrt{m})(h+\varepsilon)}{\sqrt{\alpha}} + \frac{\delta\sqrt{m}}{\sqrt{\alpha}} \tag{24}$$

holds true, where $x^{h,\delta}_{m,\alpha,n}$ *is approximate solution determined by (19).*

Proof. Using triangle's rule we obtain

$$\|x_0 - x^{h,\delta}_{m,\alpha,n}\| \leq \|x_0 - x_{m,\alpha}\| + \|x_{m,\alpha} - x^h_{m,\alpha,n}\| + \|x^h_{m,\alpha,n} - x^{h,\delta}_{m,\alpha,n}\|.$$

We consider last summand:

$$\|x^h_{m,\alpha,n} - x^{h,\delta}_{m,\alpha,n}\| = \|g_{m,\alpha}(A^*_{h,n}A_{h,n})A^*_{h,n}y -$$
$$- g_{m,\alpha}(A^*_{h,n}A_{h,n})A^*_{h,n}y_\delta\| \leq \|g_{m,\alpha}(A^*_{h,n}A_{h,n})A^*_{h,n}\| \times$$
$$\times \|y - y_\delta\| \leq \sup_{0<\lambda\leq m_K} (\sqrt{\lambda} g_{m,\alpha}(\lambda)).$$

Thus, by inequality (19) we find

$$\|x^h_{m,\alpha,n} - x^{h,\delta}_{m,\alpha,n}\| \leq \frac{\delta\sqrt{m}}{\sqrt{\alpha}}. \tag{25}$$

And finally summarizing Lemma 2, Lemma 3 and relation (25) we obtain the assertion of Theorem. □

3. The balancing principle

The balancing principle consists in choosing a value of regularization parameter α such that to balance two functions which give accuracy estimation. In our case, these functions are represented by (see (24))

$$\Phi(\alpha) := \rho(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\alpha})^{-p},$$

$$\Psi(\alpha) := \frac{\rho(m+\sqrt{m})(h+\varepsilon) + \delta\sqrt{m}}{\sqrt{\alpha}}.$$

Taking into account, that (see (16))

$$\varepsilon = \begin{cases} \delta\rho^{-1} & ,0 < h \le \delta, \\ h, & h > \delta \end{cases}$$

we can represent function $\Psi(\alpha)$ as

$$\Psi(\alpha) = \frac{\rho c_1 h + c_2\delta}{\sqrt{\alpha}},$$

where

$$c_1 = \begin{cases} m + \sqrt{m} & ,0 < h \le \delta, \\ 2(m + \sqrt{m}) & ,h > \delta \end{cases}, \quad c_2 = \begin{cases} m + 2\sqrt{m} & ,0 < h \le \delta, \\ \sqrt{m} & ,h > \delta \end{cases}.$$

Thus, we can rewrite (24) in the form

$$\|x_0 - x^{h,\delta}_{m,\alpha,n}\| \le \Phi(\alpha) + \Psi(\alpha). \tag{26}$$

Since $\phi(t) = (\underbrace{\ln ... \ln}_{K\text{-times}} \frac{1}{t})^{-p}$ is monotonously increasing function then for increasing α the function $\Phi(\alpha)$ increases. By other side, the function $\Psi(\alpha)$ is monotonously decreasing. According to behavior of functions Φ and Ψ (namely, their monotonicity and concavity) to choose a value of regularization parameter $\alpha = \widehat{\alpha}$ minimizing right-hand side of (26) we will balancing values $\Phi(\alpha)$ and $\Psi(\alpha)$, i.e.

$$\Phi(\widehat{\alpha}) = \Psi(\widehat{\alpha})$$

And, hence

$$\|x_0 - x^{h,\delta}_{m,\alpha,n}\| \le 2\Phi(\widehat{\alpha}).$$

But, since function ϕ is unknown (namely, parameter p is unknown), then such a priori choice of the best value $\widehat{\alpha}$ is impossible. Therefore in considering situation we need to make use of some a posteriori choice of α. For further studing we choice the balancing principle as such rule.

Let describe this principle according to our problem. Consider two sets

$$\Delta_N = \{\alpha_i = (q^2)^i\alpha_0, i = 1, 2, ..., N\}, \quad q > 1,$$
$$\alpha_0 = n(h + \delta)^2, \quad N : \alpha_N \asymp 1,$$

and

$$M^+(\Delta_N) = \{\alpha_i \in \Delta_N : \|x^{h,\delta}_{m,\alpha_i,n} - x^{h,\delta}_{m,\alpha_j,n}\| \le \le 4\Psi(\alpha_j), \quad j = 1, 2, ..., i\}. \tag{27}$$

Within the framework of balancing principle we take

$$\alpha = \alpha_+ := \max\{\alpha \in M^+(\Delta_N)\}. \tag{28}$$

as value of regularization parameter Moreover, consider auxiliary set

$$M(\Delta_N) := \{\alpha_i \in \Delta_N : \Phi(\alpha_i) \le \Psi(\alpha_i)\}$$

and auxiliary value

$$\alpha_* := \max\{\alpha \in M(\Delta_N)\}.$$

Without loss of generally we assume that

$$M(\Delta_N) \ne \emptyset \quad \text{and} \quad \Delta_N \setminus M(\Delta_N) \ne \emptyset.$$

And finally we can estimate closeness of exact and approximate solutions for value of regularization parameter $\alpha = \alpha_+$.

4. The main results

Theorem 2. *Assume that the regularization parameter is choosing according to (28). Then for any $x_0 \in M^K_{p,\rho}(A)$, $0 < p \le p_1$, $K = 1, 2, ...,$ the following estimate*

$$\|x_0 - x^{h,\delta}_{m,\alpha_+,n}\| \le 6q\rho(\underbrace{\ln ... \ln}_{K\text{-}times}\frac{1}{\widehat{\alpha}})^{-p}$$

is valid.

Proof. First, we show that $\alpha_* \le \alpha_+$. Due to (26), behavior of functions $\Phi(\alpha), \Psi(\alpha)$ and definition of the set $M(\Delta_N)$, for any $\alpha_j < \alpha_*$ we have

$$\|x^{h,\delta}_{m,\alpha_*,n} - x^{h,\delta}_{m,\alpha_j,n}\| \le \|x_0 - x^{h,\delta}_{m,\alpha_*,n}\| + \|x_0 - x^{h,\delta}_{m,\alpha_j,n}\| \le$$

$$\le \Phi(\alpha_*) + \Psi(\alpha_*) + \Phi(\alpha_j) + \Psi(\alpha_j) \le$$

$$\le 2\Phi(\alpha_*) + \Psi(\alpha_*) + \Psi(\alpha_j) \le$$

$$\le 3\Psi(\alpha_*) + \Psi(\alpha_j) \le 4\Psi(\alpha_j).$$

Thus, $\alpha_* \in M^+(\Delta_N)$. And, hence the inequality $\alpha_* \le \alpha_+$ holds true. Further, according to (26) for $\alpha = \alpha_*$ and also definition of sets $M^+(\Delta_N)$ and $M(\Delta_N)$ we have

$$\|x_0 - x^{h,\delta}_{m,\alpha_+,n}\| \le \|x_0 - x^{h,\delta}_{m,\alpha_*,n}\| + \|x^{h,\delta}_{m,\alpha_*,n} - x^{h,\delta}_{m,\alpha_+,n}\| \le 6\Psi(\alpha_*). \quad (29)$$

It is easy to see that from definition of function Ψ it follows

$$\Psi(q^2\alpha_*) = \frac{\rho c_1 h + c_2\delta}{\sqrt{q^2\alpha_*}} = \frac{1}{q}\frac{\rho c_1 h + c_2\delta}{\sqrt{\alpha_*}} = \frac{1}{q}\Psi(\alpha_*). \quad (30)$$

By other side, obviously $\alpha_* \le \widehat{\alpha} \le q^2\alpha_*$. According to (29) and (30) we obtain

$$\|x_0 - x^{h,\delta}_{m,\alpha_+,n}\| \le 6q\Psi(q^2\alpha_*) \le 6q\Psi(\widehat{\alpha}) =$$

$$= 6q\Phi(\widehat{\alpha}) = 6q\rho(\underbrace{\ln ... \ln}_{K\text{-times}}\frac{1}{\widehat{\alpha}})^{-p}.$$

Proof of Theorem 2 is completed. □

Theorem 3. *Let $x_0 \in M^1_{p,\rho}(A)$, $0 < p \le p_1$, and the condition of Theorem 2 is satisfies. Then for any $\delta, h > 0$ the estimate*

$$\|x_0 - x^{h,\delta}_{m,\alpha_+,n}\| \le c_p\left(\ln\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{-p}$$

holds true, where $c_p = 6q\rho\left(\frac{2p+1}{2}\right)^p$.

Proof. According to $\Phi(\widehat{\alpha}) = \Psi(\widehat{\alpha})$ we find

$$\rho \ln^{-p} \frac{1}{\widehat{\alpha}} = \frac{\rho c_1 h + c_2 \delta}{\sqrt{\widehat{\alpha}}}.$$

Then

$$\widehat{\alpha} = \left(\frac{\rho c_1 h + c_2 \delta}{\rho}\right)^2 \ln^{2p} \frac{1}{\widehat{\alpha}}.$$

As for any $x > 0$ the relation $\ln x < x$ is valid, then

$$\widehat{\alpha} \leq \left(\frac{\rho c_1 h + c_2 \delta}{\rho}\right)^2 \left(\frac{1}{\widehat{\alpha}}\right)^{2p},$$

$$\widehat{\alpha} \leq \left(\frac{\rho c_1 h + c_2 \delta}{\rho}\right)^{\frac{2}{2p+1}}.$$

Hence, due to Theorem 2 we have

$$\|x_0 - x^{h,\delta}_{m,\alpha_+,n}\| \leq 6q\rho \left(\ln \left(\frac{\rho}{\rho c_1 h + c_2 \delta}\right)^{\frac{2}{2p+1}}\right)^{-p} =$$

$$= 6q\rho \left(\frac{2p+1}{2}\right)^p \left(\ln \frac{\rho}{\rho c_1 h + c_2 \delta}\right)^{-p}.$$

Denoting $c_p = 6q\rho \left(\frac{2p+1}{2}\right)^p$, we obtain the assertion of Theorem. □

Remark 2. *In the case $p_1 = 1$ and $h = 0$ the result of Theorem 3 was obtained earlier in* [13]. *Thus, Theorem 3 generalizes result of* [13] *for any $p_1 > 0$ and $h > 0$.*

Theorem 4. *Let $x_0 \in M^K_{p,\rho}(A)$, $0 < p \leq p_1$, $K = 2, 3, \ldots$ and the condition of Theorem 2 is fulfiled. Then, for sufficiently small $h, \delta > 0$ the estimate*

$$\|x_0 - x^{h,\delta}_{m,\alpha_+,n}\| \leq c_p \left[\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{\rho}{\rho c_1 h + c_2 \delta}\right]^{-p}$$

holds true, where $c_p = 2^p 6q\rho$.

Proof. $\Phi(\widehat{\alpha}) = \Psi(\widehat{\alpha})$, then

$$\rho \left(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\widehat{\alpha}}\right)^{-p} = \frac{\rho c_1 h + c_2 \delta}{\sqrt{\widehat{\alpha}}},$$

$$\widehat{\alpha} = \left(\frac{\rho c_1 h + c_2 \delta}{\rho}\right)^2 \left(\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{1}{\widehat{\alpha}}\right)^{2p}.$$

As for any $x > \underbrace{\exp(\exp(\ldots(\exp}_{K\text{-times}}(1))))$ the inequality $\underbrace{\ln \ldots \ln}_{K\text{-times}} x < x$ is valid, then

$$\widehat{\alpha} \leq \left(\frac{\rho c_1 h + c_2 \delta}{\rho}\right)^{\frac{2}{2p+1}},$$

by that we have found the upper estimate for value of regularization parameter which theoretically minimizing accuracy.

Thus, by Theorem 2 we obtain

$$\|x_0 - x^{h,\delta}_{m,\alpha_+,n}\| \le 6q\rho \left(\underbrace{\ln\ldots\ln}_{K\text{-times}} \frac{1}{\widehat{\alpha}}\right)^{-p} \le$$

$$\le 6q\rho \left[\underbrace{\ln\ldots\ln}_{K\text{-times}} \left(\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}}\right]^{-p}.$$

Further, we will find upper-bound estimate for

$$\left[\underbrace{\ln\ldots\ln}_{K\text{-times}} \left(\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}}\right]^{-p}.$$

First, let $K = 2$, i.e. we will find upper-bound estimate for

$$\left[\ln\ln \left(\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}}\right]^{-p}.$$

Obviously, that for any fixed p, $0 < p < \infty$, there exist such $h_0, \delta_0 > 0$ that for all $0 < h \le h_0$ and $0 < \delta \le \delta_0$ the inequality

$$\left(\frac{2p+1}{2}\right)^2 \le \ln \frac{\rho}{c_1\rho h + c_2\delta}$$

is fulfiled. Whence, from monotonicity of ln it follows

$$\ln\left(\frac{2p+1}{2}\right)^2 \le \ln\ln \frac{\rho}{c_1\rho h + c_2\delta},$$

$$\ln\left(\frac{2p+1}{2}\right) \le \frac{1}{2}\ln\ln \frac{\rho}{c_1\rho h + c_2\delta}.$$

$$\ln\ln\left(\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}} = \ln\ln\frac{\rho}{\rho c_1 h + c_2\delta} - \ln\frac{2p+1}{2} \ge$$

$$\ge \frac{1}{2}\ln\ln\frac{\rho}{\rho c_1 h + c_2\delta}.$$

Hence,

$$\left[\ln\ln\left(\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}}\right]^{-p} \le 2^p \left[\ln\ln\frac{\rho}{\rho c_1 h + c_2\delta}\right]^{-p}.$$

Further, in case of arbitrary $K > 2$ we will show, that for sufficiently small $h, \delta > 0$ the inequality

$$\underbrace{\ln\ldots\ln}_{K\text{-times}}\left(\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}} \ge \frac{1}{2}\underbrace{\ln\ldots\ln}_{K\text{-times}}\left(\frac{\rho}{\rho c_1 h + c_2\delta}\right) \tag{31}$$

is fulfiled. For that reason we will carry out the proof by induction. Thus, for $K=2$ the inequality (31) was proof earlier. Let assume now, that inequality (31) is fulfiled for $K-1, \quad K \geq 3$, i.e.

$$\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \left(\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}} \geq \frac{1}{2} \underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \left(\frac{\rho}{\rho c_1 h + c_2\delta}\right).$$

Then the relation

$$\underbrace{\ln \ldots \ln}_{K\text{-times}} \left(\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}} \geq \ln \left[\frac{1}{2} \underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}\right]$$

holds true.

Further,

$$\ln \left[\frac{1}{2} \underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}\right] - \frac{1}{2}\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta} =$$

$$= \ln \left[\frac{\frac{1}{2} \underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}}{\left(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}\right)^{1/2}}\right] =$$

$$= \ln \left[\frac{1}{2}\left(\underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}\right)^{1/2}\right] > 0.$$

Hence,

$$\ln \left[\frac{1}{2} \underbrace{\ln \ldots \ln}_{(K-1)\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}\right] \geq \frac{1}{2}\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}.$$

Thus, inequality (31) holds true, then

$$\left[\underbrace{\ln \ldots \ln}_{K\text{-times}} \left(\frac{\rho}{\rho c_1 h + c_2\delta}\right)^{\frac{2}{2p+1}}\right]^{-p} \leq \left[\frac{1}{2}\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}\right]^{-p} =$$

$$= 2^p \left[\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}\right]^{-p}.$$

And it means, that due to Theorem 2

$$\|x_0 - x^{h,\delta}_{m,\alpha_+,n}\| \leq 6q\rho 2^p \left[\underbrace{\ln \ldots \ln}_{K\text{-times}} \frac{\rho}{\rho c_1 h + c_2\delta}\right]^{-p}.$$

Denoting $c_p = 2^p 6q\rho$ we complete the proof of Theorem. □

Remark 3. *In* [14] *for solving severely ill-posed problems (1)-(2) with perturbed right-hand sides* y_δ *and exactly given operators* A *a combination of standard Tikhonov regularization with Morozov's discrepancy principle was considered. This approach allows to achieve the accuracy* $O((\underbrace{\ln\dots\ln}_{K\text{-}times}\frac{1}{\delta})^{-p})$ *among the set* $M^K_{p,\rho}(A), K\in\mathbb{N}$, *of solutions. Moreover, in* [14] *the lower bound* p_0 *of possible values for parameter* p $(p > p_0 > 0)$ *was used. By other side, in Theorem 4 was shown that the strategy (14), (27), (28) of solving severely ill-posed problems guarantees the same order of accuracy on the same set* $M^K_{p,\rho}(A)$ *of solutions. But in this case the upper bound of possible values for* p $(0 < p \le p_1)$ *is used.*

Remark 4. *In* [14] *for solving problems (1) with perturbed right-hand sides only and with desired solutions from the set (2) for arbitrary* $K\in\mathbb{N}$ *was shown*

$$e(M^K_{p,\rho}(A),\delta) = O((\underbrace{\ln\dots\ln}_{K\text{-}times}\frac{1}{\delta})^{-p}),$$

where

$$e(M^K_{p,\rho}(A),\delta) := \inf_{S:Y\to X}\ \sup_{x_0\in M^K_{p,\rho}(A)}\ \sup_{y_\delta\in Y:\|y-y_\delta\|\le\delta} \|x_0 - Sy_\delta\|.$$

Hence, $e(M^K_{p,\rho}(A),\delta)$ *determines the least possible accuracy of solving (1) on the set (2) among all approximate methods* $S : Y \to X$ *constructed on perturbed data* y_δ. *It means (see Theorem (4.1)* [14]*) that the value* $O((\underbrace{\ln\dots\ln}_{K\text{-}times}\frac{1}{\delta})^{-p})$ *gives the order-optimal accuracy.*

On the other hand, it follows from Theorems 3, 4 when $h = 0$ *the received accuracy of approximate solving (1) has the representation* $O((\underbrace{\ln\dots\ln}_{K\text{-}times}\frac{1}{\delta})^{-p})$. *This, in its turn, means that in the case of exactly given operator* A *the suggested approach also provides the order-optimal accuracy of solving severely ill-posed problems.*

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