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SINGLE-OBJECTIVE LINEAR PROGRAMMING PROBLEMS WITH FUZZY COEFFICIENTS AND RESOURCES

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Resume. In this paper fuzzy single-objective linear programming problem in which both technological coefficient and resources are fuzzy with linear membership function was studied and solved. The problem was solved using the approach proposed by Gasimov R. N. and Yenilmez K. Further a real case study of a manufacturing plant and the implementation of the proposed technique is presented.

Introduction

Fuzzy multi-objective linear programming (FMOLPP) problem has its many applications in the different fields of real world. Several researchers suggested different method to solve those problems. Generally, in a single-objective linear programming problem (FSOLPP), coefficients (of objective and constraint functions) as well as constraint goals are assumed to be fixed in value. But there are many practical situations where this assumptions are not valid. These coefficients as well as constraint goals may not be well defined due to lack of information of data and/or uncertain market situations. For this reasons, the different coefficients and constraint goals may be characterized by fuzzy numbers.

The idea of fuzzy set was first proposed by Zadeh [5] , as a mean of handling uncertainty that is due to imprecision rather than to randomness. After that Bellman and Zadeh [5] proposed that a fuzzy decision might be defined as the fuzzy set, defined by the intersection of fuzzy objective and constraint goals. From this view point, Tanaka and Asai [2], Zimmermann [3] introduced fuzzy linear programming problem in fuzzy environment Gasimov and Yenilmez [6] among others, considered with all fuzzy parameters. Tong considered the single objective mathematical programming problem with fuzzy constraints. After defuzzification he solved the so-obtained crisp problem by fuzzy decisive set method proposed by Sakawa and Yano[4]. Gasimov and Yenilmez considered fuzzy linear programming problem with less than type constraints. In their paper coefficients of constraints were taken as fuzzy numbers. They solved it by fuzzy decisive set method. Lai-Hawng [7] considered FMOLPP with all parameters, having a triangular possibility distribution. They used an auxiliary model and it was solved by multi-objective linear programming methods. Chanas [1] proposed a fuzzy programming in multi-objective linear programming and it was solved by parametric approach. Zimmermann [3] proposed a fuzzy multi-criteria decision making set, defined as the intersection of all fuzzy goals and constraints.

1. Linear programming problems with fuzzy technological coefficients

We consider a linear programming problem with fuzzy technological coefficients

$$max\sum_{j=1}^{n}c_jx_j \tag{1}$$

subject to $\sum_{j=1}^{n}\widetilde{a_{ij}}x_j \leq b_i, 1 \leq i \leq m, x_j \geq 0,\ 1 \leq j \leq n.$

Assumption 1. $\widetilde{a_{ij}}$ is a fuzzy number with the following linear membership function Gasimov and Yenilmez [6]:

$$\mu_{\tilde{a}_{ij}}\ (x) = \begin{cases} 1, & \text{if} \quad x\ <\ a_{ij} \\ (a_{ij}\ +\ d_{ij}\ \ - x)/d_{ij}, & \text{if} \ \ a_{ij} \leq x <\ a_{ij}\ +\ d_{ij} \\ 0, & \text{if} \quad x \geq a_{ij}\ +\ d_{ij} \end{cases}$$

where x ∈ R and $d_{ij} > 0$ for all $1 \leq i \leq m,\ 1 \leq j \leq n$. For defuzzification of this problem, we must calculate the lower and upper bounds of the optimal values Z_l and Z_u by solving the standard linear programming problems

$$Z_1 = max\sum_{j=1}^{n}c_jx_j \tag{2}$$

subject to $\sum_{j=1}^{n}a_{ij}x_j \leq b_i, 1 \leq i \leq m, x_j \geq 0, 1 \leq j \leq n$, and

$$Z_2 = max\sum_{j=1}^{n}c_jx_j \tag{3}$$

subject to $\sum_{j=1}^{n}(a_{ij}+d_{ij})x_j \leq b_i, 1 \leq i \leq m, x_j \geq 0,\ 1 \leq j \leq n.$

The objective function takes values between Z_1 and Z_2 while technological coefficients vary between a_{ij} and $a_{ij}+d_{ij}$.

Let Z_l=min (Z_1, Z_2) and Z_u=max(Z_1, Z_2), then Z_l and Z_u are called the lower and upper bounds of the optimal values, respectively.

Assumption 2. The linear crisp problems (2) and (3) have finite optimal values. In this case the fuzzy set of optimal values, denoted G, which is a subset of R^n; is defined as Gasimov and Yenilmez [6]:

$$\mu_G\,(x) = \begin{cases} 0, & \text{if} \ \ \sum_{j=1}^{n}c_jx_j \quad < Z_l \\ (\sum_{j=1}^{n}c_jx_j - Z_l)/(Z_u - Z_l), & \text{if} \ Z_l \leq \sum_{j=1}^{n}c_jx_j < Z_u \\ 1, & \text{if} \ \ \sum_{j=1}^{n}c_jx_j \quad \geq Z_u \end{cases} . \tag{4}$$

The fuzzy set of the i-th constraint, denoted F_i , which is a subset of R^m, is defined by

$$\mu_{F_i}(x) = \begin{cases} 0, & \text{if} \quad b_i < \sum_{j=1}^{n} a_{ij}x_j \\ \dfrac{b_i - \sum_{j=1}^{n} a_{ij}x_j}{\sum_{j=1}^{n} d_{ij}x_j}, & \text{if} \sum_{j=1}^{n} a_{ij}x_j \le b_i < \sum_{j=1}^{n} (a_{ij} + d_{ij})x_j \\ 1, & \text{if} \quad b_i \ge \sum_{j=1}^{n} (a_{ij} + d_{ij})x_j \end{cases} \quad (5)$$

$1 \le i \le m$.

By using the definition of the fuzzy decision proposed by Bellman and Zadeh [5] (see also Lai and Hwang [7]), the problem (1) becomes to the following optimization problem

$$max\lambda \quad (6)$$

$$\mu_G(x) \ge \lambda, \mu_{F_i}(x) \ge \lambda, \quad 1 \le i \le m,$$

$$x \ge 0, \; 0 \le \lambda \le 1.$$

By using (4) and (5), the problem (6) can be written as

$$max\lambda \quad (7)$$

$$\lambda(Z_u - Z_l) - \sum_{j=1}^{n} c_j x_j + Z_l \le 0,$$

$$\sum_{j=1}^{n} (a_{ij} + \lambda d_{ij} x_j) - b_i \le 0, \quad 1 \le i \le m,$$

where $x_j \ge 0, \; 1 \le j \le n, \; 0 \le \lambda \le 1$.

Notice that, the constraints in problem (7) containing the cross product terms λx_j, $1 \le j \le n$, are not convex. Therefore the solution of this problem requires the special approach adopted for solving general nonconvex optimization problems.

2. Linear programming problems with fuzzy technological coefficients and fuzzy right-hand-side numbers

In this section we consider a linear programming problem with fuzzy technological coefficients and fuzzy right-hand-side numbers

$$max \sum_{j=1}^{n} c_j x_j, \quad (8)$$

$$\sum_{j=1}^{n} \widetilde{a_{ij}} x_j \le \widetilde{b_i}, 1 \le i \le m,$$

$x_j \ge 0, \; 1 \le j \le n$, and at least one $x_j > 0$.

Assumption 3. $\widetilde{a_{ij}}$ and $\widetilde{b_i}$ are fuzzy numbers with the following linear membership functions:

$$\mu_{a_{ij}}(x) = \begin{cases} 1, & \text{if } x < a_{ij} \\ (a_{ij} + d_{ij} - x)/ d_{ij}, & \text{if } a_{ij} \le x < a_{ij} + d_{ij} \\ 0, & \text{if } x \ge a_{ij} + d_{ij} \end{cases}$$

and

$$\mu_{b_i}(x) = \begin{cases} 1, & \text{if } x < b_i \\ (b_i + p_i - x)/ p_i, & \text{if } b_i \le x < b_i + p_i \\ 0, & \text{if } x \ge a_{ij} + d_{ij} \end{cases},$$

$$1 \le i \le m.$$

We first calculate the lower and upper bounds of the optimal values. The optimal values Z_l and Z_u can be defined by solving the following standard linear programming problems, for which we assume that all they have the finite optimal values,

$$Z_1 = max\sum_{j=1}^{n} c_j x_j, \tag{9}$$

$$\sum_{j=1}^{m} a_{ij} x_j \le b_i,\ 1 \le i \le m, x_j \ge 0,$$

$$Z_2 = max\sum_{j=1}^{n} c_j x_j, \tag{10}$$

$$\sum_{j=1}^{n} (a_{ij} + d_{ij})\, x_j \le b_i, 1 \le i \le m, x_j \ge 0,$$

$$Z_3 = max\sum_{j=1}^{n} c_j x_j, \tag{11}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i + p_i, 1 \le i \le m, x_j \ge 0,$$

$$Z_4 = max\sum_{j=1}^{n} c_j x_j\ , \tag{12}$$

$$\sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j \le b_i + p_i,\ 1 \le i \le m, x_j \ge 0.$$

Let Z_l=min(Z_1,Z_2,Z_3,Z_4) and Z_u = max(Z_1,Z_2,Z_3,Z_4). The objective function takes values between Z_l and Z_u while technological coefficients take values between a_{ij} and $a_{ij} + d_{ij}$ and the right-hand-side numbers take values between b_i and $b_i + p_i$.

Then, the fuzzy set of optimal values denoted as early G, which is a subset of R^n is defined by

$$\mu_G(x) = \begin{cases} 0, & \text{if } \sum_{j=1}^{n} c_j x_j < Z_l \\ (\sum_{j=1}^{n} c_j x_j - Z_l)/(Z_u - Z_l) & \text{if } Z_l \le \sum_{j=1}^{n} c_j x_j < Z_u \\ 1, & \text{if } \sum_{j=1}^{n} c_j x_j \ge Z_u \end{cases} . \quad (13)$$

The fuzzy set of the i-th constraint denoted F_i, which is a subset of R^n is defined by

$$\mu_{F_i}(x) = \begin{cases} 0, & \text{if } b_i < \sum_{j=1}^{n} a_{ij} x_j \\ \dfrac{b_i - \sum_{j=1}^{n} a_{ij} x_j}{\sum_{j=1}^{n} d_{ij} x_j + p_i}, & \text{if } \sum_{j=1}^{n} a_{ij} x_j \le b_i < \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i, \\ 1, & \text{if } b_i \ge \sum_{j=1}^{n} (a_{ij} + d_{ij}) x_j + p_i \end{cases}$$

$$1 \le i \le m. \quad (14)$$

Then, by using the method of defuzzification for the problem (1), the problem (8) is reduced to the following crisp problem:

$$max\lambda \quad (15)$$

$$\lambda(Z_u - Z_l) - \sum_{J=1}^{n} c_j x_j + Z_l \le 0$$

$$\sum_{j=1}^{n} (a_{ij} + \lambda d_{ij}) x_j + \lambda p_i - b_i \le 0, 1 \le i \le m, \quad x_j \ge 0,\ 0 \le \lambda \le 1.$$

Notice that, the problem (15) is also a nonconvex programming problem, similar to the problem (7).

3. The algorithm of the fuzzy decisive set method

This method is based on the idea that, for a fixed value of λ, the problems (7) and (15) are linear programming problems. Obtaining the optimal solution λ^* to the problems (7) and (15) is equivalent to determining the maximum value of λ so that the feasible set is nonempty.

The algorithm of this method for the problem (7) is presented below. The algorithm for the problem (15) is similar.

Step 1. Set $\lambda = 1$ and test whether a feasible set satisfying the constraints of the problem (7) exists or not using phase one of the simplex method. If a feasible set exists, set $\lambda = 1$. Otherwise, set $\lambda^L = 0$ and $\lambda^R = 1$ and go to the next step.

Step 2. For the value of $\lambda = (\lambda^L + \lambda^R)/2$, update the value of λ^L and λ^R using the bisection method as follows :

$\lambda^L = \lambda$ if feasible set is nonempty for λ,

$\lambda^R = \lambda$ if feasible set is empty for λ.

Consequently, for each λ, test whether a feasible set of the problem (7) exists or not using phase one of the Simplex method and determine the maximum value λ^* satisfying the constraints of the problem (7).

4. A numerical example produces and transports problem

The operations of a concrete manufacturing plant, which produces and transports concrete to building sites, have been analyzed. Fresh concrete is produced at a central concrete plant and transported by seven transit mixers over the distance ranging 1500–3000 m (depending on the location of the construction site) to the three construction sites.

Concrete pumps and interior vibrators are used for delivering, placing and consolidating the concrete at each construction site. Table 1 illustrates the manufacturing capacities of the plant, operational capacity of the concrete mixer, interior vibrator, pumps and manpower requirement at the three construction sites. A quick analysis will reveal the complexity of the variables and constraints of this concrete production plant and delivery system. The plant manager's task will be to optimize the profit by utilizing the maximum plant capacity while meeting the three-construction site's concrete and other resource requirement through a feasible schedule.

Table 1. Concrete plant capacity and construction site's resource demands

	Site A	Site B	Site C	Capacity	Tolerance (resources)	Tolerance (coefficients)		
Transit mixers	1 m^3/h	1 m^3/h	1 m^3/h	15	5	1	1	1
Worker requirement	7	5	3	80	40	4	3	1
Concrete pumps	3 m^3/h	4.4 m^3/h	10 m^3/h	100	30	1	2	4
(Tolerance)	47 m^3	60 m^3	72 m^3	-	-			

4.1. Objective formulation

Success of any decision model will directly depend on the formulation of the objective function taking into account all the influential factors. We modeled the final objective function taking into account factor profit expressed as $\$/m^3$.

Profit: The expected profit as related to the volume of concrete to be manufactured is modeled as the first objective and is shown in Table 2.

Table 2. Modeling profit as an objective

	Site A	Site B	Site C
Expected profit (AU$/$m^3$)	10	11	15

4.2. Variables that optimize the objective function

After knowing the objective function the next task is to determine the variables that optimizes the objective function. In our problem it is to find the optimal value of unknowns $x_j, j = 1, 2, 3$, that represent quantities of concrete which have to be delivered to Site A, B and C respectively and corresponding optimal values of the objective function Z.

According to problem requirements and available data (see Table 1 and Table 2), the model formulated as

$$maxZ = 10x_1 + 11x_2 + 15x_3 \tag{16}$$

subject to

$$\tilde{1}x_1 + \tilde{1}x_2 + \tilde{1}\ x_3 \leq \widetilde{15},$$
$$\tilde{7}x_1 + \tilde{5}x_2 + \tilde{3}x_3 \leq \widetilde{80},$$
$$\tilde{3}x_1 + \widetilde{4.4}x_2 + \widetilde{10}x_3 \leq \widetilde{100},$$

$x_j \geq 0, j = 1, 2, 3,$

$$(a_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ 7 & 5 & 3 \\ 3 & 4.4 & 10 \end{pmatrix}, (d_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}, (a_{ij} + d_{ij}) = \begin{pmatrix} 2 & 2 & 2 \\ 11 & 8 & 4 \\ 4 & 6.4 & 14 \end{pmatrix}$$

$$(b_i) = \begin{pmatrix} 15 \\ 80 \\ 100 \end{pmatrix}, (p_i) = \begin{pmatrix} 5 \\ 40 \\ 30 \end{pmatrix}, (b_i + p_i) = \begin{pmatrix} 20 \\ 120 \\ 130 \end{pmatrix},\ i = 1, 2, 3, j = 1, 2, 3.$$

We shall apply our algorithm for this fuzzy task (16), calculate the lower and upper bounds of the optimal values. The bounds of the optimal values Z_l^* and Z_u^* are obtained by solving the standard linear programming problems

$$Z_1 = maxZ = 10x_1 + 11x_2 + 15x_3 \tag{17}$$

subject to

$$x_1 + x_2 + x_3 \leq 15$$
$$7x_1 + 5x_2 + 3x_3 \leq 80$$
$$3x_1 + 4.4x_2 + 10x_3 \leq 100$$
$$x_j \geq 0,\ j = 1, 2, 3,$$

$$Z2 = maxZ = 10x_1 + 11x_2 + 15x_3 \tag{18}$$

subject to

$$x_1 + x_2 + x_3 \leq 20$$
$$7x_1 + 5x_2 + 3x_3 \leq 120$$
$$3x_1 + 4.4x_2 + 10x_3 \leq 130$$
$$x_j \geq 0,\ j = 1, 2, 3,$$

$$Z_3 = maxZ = 10x_1 + 11x_2 + 15x_3 \tag{19}$$

subject to

$$2x_1 + 2x_2 + 2x_3 \subset\leq 15$$
$$11x_1 + 8x_2 + 4x_3 \leq 80$$
$$4x_1 + 6.4x_2 + 14x_3 \leq 100$$
$$x_j \geq 0,\ j = 1, 2, 3,$$

$$Z_4 = maxZ = 10x_1 + 11x_2 + 15x_3 \tag{20}$$

subject to

$$2x_1 + 2x_2 + 2x_3 \leq 20$$
$$11x_1 + 8x_2 + 4x_3 \leq 120$$
$$4x_1 + 6.4x_2 + 14x_3 \leq 130$$

$$x_j \geq 0,\ j = 1, 2, 3.$$

Optimal value of these problems are Z=(189.3,250,110,145), and therefore $Z_l^* = 110$, $Z_u^* = 250$.

By using these optimal values, the problem (16) can be reduced by the following non-linear programming problem:

$$\max \lambda$$
$$(10x_1 + 11x_2 + 15x_3 - 110)/(250 - 110) \geq \lambda,$$
$$(15 - (x_1 + x_2 + x_3))/(2x_1 + 2x_2 + 2x_3 + 5) \geq \lambda,$$
$$(80 - (7x_1 + 5x_2 + 3x_3))/(4x_1 + 3x_2 + x_3 + 40) \geq \lambda,$$
$$(100 - (3x_1 + 4.4x_2 + 10x_3))/(x_1 + 2x_2 + 4x_3 + 30) \geq \lambda,$$
$$x_j \geq 0,\ j = 1, 2, 3, \quad 0 \leq \lambda \leq 1.$$

That is,

$$max\lambda \tag{21}$$
$$10x_1 + 11x_2 + 15x_3 \geq 140\lambda + 110,$$
$$(2\lambda + 1)\,x_1 + (2\lambda + 1)\,x_2 + (2\lambda + 1)\,x_3 \geq 15 - 5\lambda,$$
$$(4\lambda + 7)\,x_1 + (3\lambda + 5)\,x_2 + (\lambda + 3)\,x_3 \geq 80 - 40\lambda,$$
$$(\lambda + 3)\,x_1 + (2\lambda + 4.4)\,x_2 + (4\lambda + 10)\,x_3 \geq 100 - 30\lambda,$$
$$x_j \geq 0,\ j = 1, 2, 3, \quad 0 \leq \lambda \leq 1.$$

Let us solve the problem (21) by using fuzzy decisive set method. For λ=1, the problem can be written as

$$10x_1 + 11x_2 + 15x_3 \geq 250,$$
$$3x_1 + 3x_2 + 3x_3 \leq 10,$$
$$11x_1 + 8x_2 + 4x_3 \leq 40,$$
$$4x_1 + 6.4x_2 + 14x_3 \leq 70,$$
$$x_j \geq 0,\ j = 1, 2, 3 \quad .$$

Since the feasible set is empty, by taking λ^L=0 and $\lambda^R = 1$, we applied the algorithm that explained in subsection 3 and obtained:

$$\lambda = 0.2188; \lambda = 0.2031; \lambda = 0.2109; \lambda = 0.2070; \lambda = 0.2089;$$
$$\lambda = 0.2080; \lambda = 0.2085; \lambda = 0.2083; \lambda = 0.2081; \lambda = 0.2081.$$

Consequently, we obtain the optimal value of λ at the fifteenth iteration by using the fuzzy decisive set method. The optimal solution is x_1^*=1.67, x_2^*=0, x_3^*=80.12, $Z^* = 139.1$ and $\lambda^* = 0.2081$.

Conclusion

In this paper, fuzzy single-objective linear programming problem in which both the resources and the technological coefficients are fuzzy with linear membership function was studied the problem was solved by fuzzy decisive set method. This procedure may be very helpful for any fuzzy single-criteria decision making problem.

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