Журнал обчислювальної та прикладної математики | 2016 № 2 (122) | Journal of Computational & Applied Mathematics

UDC 519.642

ON OPTIMAL SELECTION OF GALERKIN'S INFORMATION FOR SOLVING SEVERELY ILL-POSED PROBLEMS

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Резюме. Для розв'язування експоненційно некоректних задач розроблено економічний проекційний метод, який полягає у комбінуванні стандартного метода Тіхонова та принципу нев'язки Морозова. При цьому встановлено, що запропонований алгоритм забезпечує оптимальний порядок інформаційної складності на класі досліджуваних задач.

Abstract. An economical projection method is developed for solving exponentially ill-posed problems. The method consist in combination of the standard Tikhonov method and the Morozov discrepancy principle. Herewith, it is established that this approach provides optimal order of information complexity on the class of problems under consideration.

1. Introduction

The implicit (a posteriori) choice of the regularization parameter without any information on smoothness of a desired solution is usually assume to be the key issue in the theory of ill-posed problems. It is well-known, there are a lot of different rules of a regularization parameter choice among them we mention discrepancy principle [6, 8, 9, 20], Gfrerer's method [3, 19], the monotone error rule [27], the balancing principle [2, 4, 14, 25] which sometimes is called the Lepskij principle. Nowadays, it is sure the discrepancy principle is the most common one.

In the present paper that is extension of the research started in [23, 24] the authors develop economical projection method for effective solving severely ill-posed problems. As a regularization the standard Tikhonov method is applied. Unlike to above-mentioned works, the regularization parameter is chosen a posteriori, namely, according with the balancing principle. Moreover, it is established that a proposed strategy maintains optimal oder accuracy on the class of problems under consideration, as well as provides oder estimates of the information complexity.

The organization of the material is as follows: in Section 2 we give the statement of the problem. Further in Section 3 the regularization and discretization methods are described. Auxiliary statements and facts are in Section 4. An algorithm of the regularization parameter choice by discrepancy principle is

Key words. Severely ill-posed problems; minimal radius of Galerkin information; discrapency principle; information complexity.

presented in Section 5. The combination of proposed methods allows to established optimal order accuracy for solving equations from the class of problems under research. Finally, in Section 6, the authors establish the main result. Namely, the order estimate for the minimal radius of the Galerkin information is obtained.

2. Statement of the problem

Following [23] we present the rough statement of the problem. Consider Fredholm's integral equation of the first kind

$$Ax(t) = f(t), \qquad t \in [0,1], \tag{1}$$

with

$$Ax(t) = \int_0^1 a(t,\tau)x(\tau)d\tau, \tag{2}$$

acting continuously in $L_2 = L_2(0,1)$. Suppose that $\mathrm{Range}(A)$ is not closed in L_2 and $f \in \mathrm{Range}(A)$.

We also assume that a perturbation $f_\delta \in L_2 : \|f - f_\delta\| \le \delta, \quad \delta > 0$ is given instead of the right-hand side of the equation (1).

The problem (1) is regarded as severely ill-posed problem if its solution has substantially worse smoothness than a kernel $a(\cdot,\tau)$ In such case it is nature to assume that an exact solutions satisfies some logarithmic source condition, in other words it belongs to the set

$$M_p(A) := \{u : u = \ln^{-p}(A^*A)^{-1}v, \quad \|v\| \le \rho\},$$

where p, ρ are some positive parameters and A^* is adjoined operator to A. Such problems are called exponentially ill-posed (see e.g. [5]).

Note, that the exact information about smoothness, namely, the parameter p, is usually not available by practical experiment. For this reason the set

$$M(A) := \bigcup_{p\in(0,p_1]} M_p(A) \tag{3}$$

is considered in place of $M_p(A)$. Here $p_1 < \infty$ is an upper bound for possible values of p.

Within the framework of our researches we construct an approximation to the exact solution $x^\dagger$ (1), which has minimal norm in L_2 and belongs to the set $M(A)$. From now on, we assume that a parameter p is unknown.

Let $\{e_i\}_{i=1}^\infty$ be some orthonormal basis in L_2, and let P_m denotes the orthogonal projection onto $\mathrm{span}\{e_1, e_2, \dots, e_m\}$

$$P_m\varphi(t) = \sum_{i=1}^m (\varphi, e_i)e_i(t).$$

Consider the following class of operators (2):

$$\mathcal{H}_\gamma^{r,s} = \{A : \|A\| \le \gamma_0, \quad \sum_{n+m=1}^\infty \hat{a}_{n,m}^2\, \underline{n}^{2r}\underline{m}^{2s} \le \gamma_1^2\}, \qquad r,s > 0, \tag{4}$$

where

$$\hat{a}_{n,m} = \int_0^1 \int_0^1 e_n(t) a(t,\tau) e_m(\tau) d\tau dt,$$

$\gamma_0 \le e^{-\frac{1}{2}}, \gamma = (\gamma_0; \gamma_1)$, $\underline{n} = 1$ if $n = 0$ and $\underline{n} = n$ otherwise.

If the kernel $a(t,\tau)$ of A has mixed partial derivatives and the inequalities

$$\int_0^1 \int_0^1 \left[\frac{\partial^{i+j} a(t;\tau)}{\partial t^i \partial \tau^j} \right]^2 dt d\tau < \infty$$

hold for all $i = 0, 1, \dots, r$, $j = 0, 1, \dots, s$ then it is known (see e.g. [16]), $A \in \mathcal{H}_\gamma^{r,s}$ for some $\gamma = (\gamma_0, \gamma_1)$.

From now on, class of equations (1) with operators belonging to $\mathcal{H}_\gamma^{r,s}$ (4) and solutions from $M(A)$ (3) will be denoted by $(\mathcal{H}_\gamma^{r,s}, M(A))$. In the present paper we concentrate on the study of projection methods for solving equations belonging to $(\mathcal{H}_\gamma^{r,s}, M(A)), \quad r \ge s$.

A discretization projection scheme of equations (1) with the perturbed right-hand side one can define by means of a finite set of the inner products

$$(Ae_j, e_i), \qquad (i,j) \in \Omega, \tag{5}$$

$$(f_\delta, e_k), \qquad k \in \omega_1, \quad \omega_1 = \{i \colon (i,j) \in \Omega\}, \tag{6}$$

where Ω to be an bounded domain of the coordinate plane $[1,\infty) \times [1,\infty)$. The inner products (5), (6) are used to call the Galerkin information about (1). Here $\mathrm{card}(\Omega)$ is the total number of the inner products (5). In particular, if $\Omega = [1,n] \times [1,m]$, then one deal with the standard Galerkin discretization scheme, $\mathrm{card}(\Omega) = n \cdot m$. Researches for various classes of ill-posed problems related to such scheme of discretization were conducted in a number of works among which we mention [7, 17, 18].

Definition 11. A projection method of solving (1) can be associated with any mapping $\mathcal{P} = \mathcal{P}(\Omega) : L_2 \to L_2$ which by the Galerkin information (5), (6) about (1) provides a correspondence between the right-hand side of the equation being solved and an element $\mathcal{P}(A_\Omega) f_\delta \in L_2$, which is a polynomial by the basis $\{e_i\}_{i=1}^\infty$ with harmonic numbers from $\omega_2 := \{j \colon (i,j) \in \Omega\}$. This element is taken as an approximate solution (1).

The error of the method $\mathcal{P}(\Omega)$ on the class of equations $(\mathcal{H}_\gamma^{r,s}, M_p(A))$ is defined as

$$e_\delta\left(\mathcal{H}_\gamma^{r,s}, M(A), \mathcal{P}(\Omega)\right) = \sup_{A \in \mathcal{H}_\gamma^{r,s}} \sup_{x^\dagger \in M(A)} \sup_{f_\delta : \|f - f_\delta\| \le \delta} \|x^\dagger - \mathcal{P}(A_\Omega) f_\delta\|.$$

The minimal radius of the Galerkin information is given by

$$R_{N,\delta}\left(\mathcal{H}_\gamma^{r,s}, M(A)\right) = \inf_{\Omega \colon \mathrm{card}(\Omega) \le N} \inf_{\mathcal{P}(\Omega)} e_\delta\left(\mathcal{H}_\gamma^{r,s}, M(A), \mathcal{P}(\Omega)\right).$$

This value describes the minimal possible accuracy (among all projection methods), while the Galerkin information amount are bound. Thus, $R_{N,\delta}$ characterizes information complexity on the class of problems $(\mathcal{H}_\gamma^{r,s}, M(A))$.

It is easy to see, that such studies belong to the range of problems from Information Based Complexity Theory. The fundamentals of this theory were introduced in monographs [28, 29]. It should be noted that in recent years the interest to such researches in the light of ill-posed problems is greatly increase. In the work [18] first economical projection methods for solving moderately ill-posed problems were constructed. The standard Galerkin scheme was employed as discretization scheme. But first order estimates for complexity of moderately ill-posed problems were obtained in [16, 21, 22]. The authors point to the fact that optimal orders of such values are achieved under a modified Galerkin scheme that is called hyperbolic cross. The complexity of severely ill-posed problems began to be study relatively recently. These researches are highlighted in the series of works, we mention [7, 23, 24].

In the present paper as opposite to above-mentioned one, an economical projection scheme with a posteriori rule of regularization parameter choice will be developed for solving severely ill-posed problems.

3. Regularization and discretization strategies

To guarantee stable approximations we apply the standard Tikhonov method. By means of this method the rugularized solution x_α is defined as the solution of the variation problem

$$I_\alpha(x) := \|Ax - f_\delta\|^2 + \alpha\|x\|^2 \to \min. \tag{7}$$

For a numerical realization of the standard Tikhonov method it is necessary to carry out all computations with finite amount of input data. For that reason the variation problem (7) is replaced by following

$$I_{\alpha,n}(x) = \|A_n x - f_\delta\|^2 + \alpha\|x\|^2 \to \min,$$

where A_n is some operator of the finite rank.

The idea to apply the hyperbolic cross to operator equations of the second kind belongs to S.V. Pereverzev and implements in the series of works (see e.g. [10–13]). The efficiency of the hyperbolic cross for ill-posed problems has been demonstrated in [15, 16, 23]. Within the framework of our researches we apply a projection scheme with $\Omega = \Gamma_n^a$, where

$$\Gamma_n^a = \{1\} \times [1; 2^{2an}] \bigcup_{k=1}^{2n} (2^{k-1}; 2^k] \times [1; 2^{(2n-k)a}] \subset [1; 2^{2n}] \times [1; 2^{2an}] \tag{8}$$

is a hyperbolic cross on the coordinate plane by the basis $\{e_i\}_{i=1}^\infty$ involved in the definition of the class $\mathcal{H}_\gamma^{r,s}$. Here for $r > s$ the parameter a is an arbitrary real number such that $1 < a < \frac{r}{s}$, and for $a = 1$ we set $r = s$. To simplify computations we assume that ak are integer numbers. An approximate solution one can find from an operator equation of the second kind

$$\alpha x + A_n^* A_n x = A_n^* f_\delta.$$

On other words, we seek an approximate solution $x = x_{\alpha,n}^\delta$ of the form

$$x_{\alpha,n}^\delta = g_\alpha(A_n^* A_n) A_n^* f_\delta, \tag{9}$$

where $g_\alpha(\lambda) = (\alpha + \lambda)^{-1}$, and

$$A_n = P_1 A P_{2^{2an}} + \sum_{k=1}^{2n} (P_{2^k} - P_{2^{k-1}}) \, A P_{2^{(2n-k)a}}. \tag{10}$$

Moreover we introduce following auxiliary elements

$$x_\alpha = g_\alpha(A^*A)A^*f, \tag{11}$$

$$x_{\alpha,n} = g_\alpha(A_n^*A_n)A_n^*f. \tag{12}$$

4. Auxiliary results

In this Section we formulate some definitions and facts, and also the series of auxiliary assertions which shell later need.

It is well-known (see e.g. [30]), that for any linear bounded operator A the inequalities

$$\begin{gathered} \|(\alpha I + A^*A)^{-1}\| \le \alpha^{-1}, \quad \|(\alpha I + A^*A)^{-1}A^*\| \le \tfrac{1}{2\sqrt{\alpha}}, \\ \|A(\alpha I + A^*A)^{-1}A^*\| \le 1 \end{gathered} \tag{13}$$

hold.

Lemma 1. *(see [30, p. 34]) If g to be bounded Borel measurable function on $[0;\gamma_0^2]$, $\quad A \in \mathcal{L}(L_2, L_2)$, $\quad \|A\| \le \gamma_0$, then*

$$\begin{gathered} A^*g(AA^*) = g(A^*A)A^*, \\ Ag(A^*A) = g(AA^*)A. \end{gathered} \tag{14}$$

Lemma 2. *(see [20]) Let $\|A\| \le \gamma_0 \le e^{-1/2}$. Then for sufficiently small $\alpha \in (0, e^{-2p})$ it holds*

$$\|Ax_\alpha - f\| \le \gamma_0^{-1}\rho\sqrt{\alpha}\ln^{-p} 1/\alpha,$$

where x_α is determined by (11).

Lemma 3. *(see [20]) Let $\|A\| \le \gamma_0 \le e^{-1/2}$, and α is such that*

$$\|Ax_\alpha - f\| \le d' \delta,$$

where $d' > 0$ is some positive constant. Then the estimate

$$\|x^\dagger - x_\alpha\| \le \xi \ln^{-p} 1/\delta$$

is fulfilled. The constant $\xi > 0$ depends only on d', ρ and p.

Lemma 4. *For any $\alpha > 0$ and $n \in \mathbb{N}$ the estimate*

$$\|Ax_\alpha - f\| \le \|A_n x_{\alpha,n}^\delta - P_{2^{2n}} f_\delta\| + \left(\|(I - P_{2^{2n}})f\|^2 + \delta^2\right)^{1/2} + \frac{5}{4}\rho\|A - A_n\|$$

holds, where x_α and $x_{\alpha,n}^\delta$ is determined by (11) and (9), respectively.

Proof. First off all, we note that

$$\|x^\dagger\| = \| \ln^{-p}(A^*A)v\| \le \rho \sup_{0<\lambda\le\gamma_0^2} |\ln^{-p} 1/\lambda| \le \rho. \tag{15}$$

Further, consider the decomposition

$$Ax_\alpha - f = A_n x^\delta_{\alpha,n} - P_{2^{2n}} f_\delta + S_1 + S_2,$$

where

$$S_1 := -\left(I - A_n g_\alpha(A_n^* A_n) A_n^*\right)\left(f - P_{2^{2n}} f_\delta\right),$$
$$S_2 := \left(A g_\alpha(A^*A)A^* - A_n g_\alpha(A_n^* A_n) A_n^*\right) f.$$

Now we are going to bound each term S_1, S_2. By (13), (14) we immediate find

$$\begin{aligned}\|S_1\| &\le \|I - A_n(\alpha I + A_n^*A_n)^{-1}A_n^*\|\|f - P_{2^{2n}} f_\delta\| \le \\ &\le \|I - (\alpha I + A_nA_n^*)^{-1}A_nA_n^*\|\|(I - P_{2^{2n}})f + P_{2^{2n}}(f - f_\delta)\| \le \\ &\le \left(\|\left(I - P_{2^{2n}}\right) f\|^2 + \delta^2\right)^{\frac{1}{2}}.\end{aligned}$$

It remains to estimate the norm of S_2. First, rewrite S_2 as follows

$$\begin{aligned}S_2 &= \left(A g_\alpha(A^*A)A^* - A_n g_\alpha(A_n^*A_n)A_n^*\right) f = \\ &= \alpha\left(\alpha I + A_nA_n^*\right)^{-1}\left(AA^* - A_nA_n^*\right)\left(\alpha I + AA^*\right)^{-1} f = \overline{s}_1 + \overline{s}_2,\end{aligned}$$

where

$$\overline{s}_1 := \alpha\left(\alpha I + A_nA_n^*\right)^{-1}\left(A - A_n\right)A^*\left(\alpha I + AA^*\right)^{-1} Ax^\dagger,$$
$$\overline{s}_2 := \alpha\left(\alpha I + A_nA_n^*\right)^{-1} A_n\left(A^* - A_n^*\right)\left(\alpha I + AA^*\right)^{-1} Ax^\dagger.$$

Further, we bound norms of $\overline{s}_1$ and $\overline{s}_2$. By (13), (14) and (15) we obtain

$$\begin{aligned}\|\overline{s}_1\| &\le \alpha\|\left(\alpha I + A_nA_n^*\right)^{-1}\|\|A - A_n\|\|\left(\alpha I + A^*A\right)^{-1} A^*A\|\|x^\dagger\| \le \\ &\le \rho\|A - A_n\|, \\ \|\overline{s}_2\| &\le \alpha\|\left(\alpha I + A_nA_n^*\right)^{-1} A_n\|\|A^* - A_n^*\|\|\left(\alpha I + AA^*\right)^{-1} A\|\|x^\dagger\| \le \\ &\le \frac{\rho}{4}\|A - A_n\|.\end{aligned}$$

Thus,

$$\|S_2\| \le \|\overline{s}_1\| + \|\overline{s}_2\| \le \frac{5\rho}{4}\|A - A_n\|.$$

Summing up the above bounds, we finally get

$$\begin{aligned}\|Ax_\alpha - f\| &\le \|A_n x^\delta_{\alpha,n} - P_{2^{2n}} f_\delta\| + \\ &\quad + \left(\|(I - P_{2^{2n}})f\|^2 + \delta^2\right)^{1/2} + \frac{5\rho}{4}\|A - A_n\|.\end{aligned}$$

The lemma is proved. □

Lemma 5. *The two-side estimates*

$$\begin{aligned}2^{2n} n < \mathrm{card}(\Gamma_n^1) &\le 2 \cdot 2^{2n} n, \qquad r = s, \\ \eta_1 2^{2an} \le \mathrm{card}(\Gamma_n^a) &\le \eta_2 2^{2an}, \qquad r > s,\end{aligned} \tag{16}$$

are hold, with $\eta_1 = 1 + \frac{1-2^{3(1-a)}}{1-2^{1-a}}$, $\eta_2 = \frac{2-2^{1-a}}{1-2^{1-a}}$.

Proof. From (8) it follows

$$\mathrm{card}(\Gamma_n^a) = \sum_{k=0}^{2n} \mathrm{card}(Q_k),$$

where

$$Q_k = \begin{cases} (2^{k-1}; 2^k] \times [1; 2^{(2n-k)a}], & k = 1, 2, \dots, 2n \\ \{1\} \times [1; 2^{2an}], & k = 0 \end{cases},$$

and we obtain

$$\mathrm{card}(\Gamma_n^a) = 2^{2an} + \frac{1}{2} \sum_{k=1}^{2n} 2^k 2^{(2n-k)a}.$$

Further, consider two cases. It is obvious that for $r = s$ it holds

$$\mathrm{card}(\Gamma_n^1) = 2^{2n} + \frac{1}{2} \sum_{k=1}^{2n} 2^{2n} = 2^{2n}(1+n) = 2^{2n} n \left(1 + \frac{1}{n}\right).$$

Hence,

$$2^{2n} n < \mathrm{card}(\Gamma_n^1) \leq 2 \cdot 2^{2n} n.$$

When $r > s$ the sequence $\{\mathrm{card}(Q_k)\}_{k=1}^{2n}$ is the geometric progression with the quotient 2^{1-a}, and the relation

$$\mathrm{card}(\Gamma_n^a) = 2^{2an} \left(1 + \frac{1}{2} \sum_{k=1}^{2n} 2^{k(1-a)}\right)$$

is hold. It follows that

$$\mathrm{card}(\Gamma_n^a) = \frac{1}{2} 2^{2an} \left(1 + \sum_{k=0}^{2n} 2^{k(1-a)}\right) = \frac{1}{2} 2^{2an} \left(1 + \frac{1 - 2^{(1-a)(2n+1)}}{1 - 2^{(1-a)}}\right).$$

Further, we obtain lower and upper bounds for the bracketed expression:

$$1 + \frac{1 - 2^{(1-a)(2n+1)}}{1 - 2^{(1-a)}} = \frac{2 - 2^{1-a}\left(1 + 2^{(1-a)2n}\right)}{1 - 2^{1-a}} \leq \frac{2 - 2^{1-a}}{1 - 2^{1-a}},$$

$$1 + \frac{1 - 2^{(1-a)(2n+1)}}{1 - 2^{(1-a)}} \geq 1 + \frac{1 - 2^{3(1-a)}}{1 - 2^{1-a}}.$$

Thus, finally we get

$$\left(1 + \frac{1 - 2^{3(1-a)}}{1 - 2^{1-a}}\right) 2^{2an} \leq \mathrm{card}(\Gamma_n^a) \leq \frac{2 - 2^{1-a}}{1 - 2^{1-a}} 2^{2an}.$$

The statement of the lemma is proved. □

It is known (see. [21]), that for any $A \in \mathcal{H}_\gamma^{r,s}$ the inequality

$$\|A - A_n\| \leq \varepsilon_{r,s}(n) \tag{17}$$

is fulfilled, where

$$\varepsilon_{r,s}(n) = \begin{cases} \gamma_1 2^{r+1/2}\sqrt{n}2^{-2rn}, & r = s \\ \gamma_1 \left(1 + \frac{2^r}{1-2^{as-r}}\right) 2^{-2nas}, & r > s \end{cases}.$$

5. Error estimate of the algorithm

5.1. **Algorithm (Discrepancy principle as stop rule).** Let us fix $\theta \in (0,1)$ and $\alpha_0 \in (0,1]$. We are going to choose the regularization parameter α according with the rule

$$\alpha \in \Delta_\theta(\delta) = \{ \quad \alpha\colon \alpha = \alpha_m := \alpha_0\theta^m, \quad m = 0,1,2,\dots, \quad \alpha \in (\delta^2, \alpha_0]\}, \tag{18}$$

and the discretization parameter n as follows

$$\varepsilon_{r,s}(n) = \frac{4}{5\rho}\delta. \tag{19}$$

Now, we describe proposed algorithm with the discrepancy principle as a stop rule concerning to studied problem.

1. Input data: $A \in \mathcal{H}_\gamma^{r,s}$, f_δ, δ, ρ.
2. To construct A_n (10) and $P_{2^{2n}} f_\delta$ we compute the inner products (5), (6).
3. The cycle: $\quad m = 1,2,\dots,M, \qquad \alpha = \alpha_m = \alpha_0\theta^m$.
   An approximate solution $x^\delta_{\alpha_m,n}$ (9) is computed by solving the equation
   $$\alpha_m x^\delta_{\alpha_m,n} + A_n^* A_n x^\delta_{\alpha_m,n} = A_n A^* f_\delta.$$
   The cycle is running as long as stop rule conditions will be meet.
4. The stop rule (the discrepancy principle)
   $$\|A_n x^\delta_{\alpha_M,n} - P_{2^{2n}} f_\delta\| \le d\delta, \tag{20}$$
   $$\|A_n x^\delta_{\alpha_m,n} - P_{2^{2n}} f_\delta\| > d\delta, \tag{21}$$
   with $m < M$, $\quad d > \sqrt{2}+1$, and $x^\delta_{\alpha_M,n}$ is determined by (9).

Introduced projection method (10), (18)–(21) we denoted as $\mathcal{P}$‘.

Lemma 6. *Let α_M such that the conditions (20) and (21) are satisfied with $d > \sqrt{2}+1$, and the parameter n in (10) is chosen as (19). Then there are the constants $d_1, d_2 > 0$, that the two-side estimate*

$$d_1\delta \le \|Ax_{\alpha_M} - f\| \le d_2\delta$$

is fulfilled.

Proof. First, note that by (17) and (19) it holds

$$\begin{aligned} \frac{5\rho}{4}\|A - A_n\| &\le \delta, \\ \|(I - P_{2^{2n}})f\| &\le \delta. \end{aligned} \tag{22}$$

If α_M meets the condition (20) then

$$\|A_n g_{\alpha_M}(A_n^* A_n) A_n^* f_\delta - P_{2^{2n}} f_\delta\| \le d\delta,$$

and applying Lemma 4 we obtain

$$\|Ax_{\alpha_M} - f\| \le d\delta + \sqrt{2\delta^2} + \delta = (d + \sqrt{2} + 1)\delta.$$

At the same time, kipping in mind (21), for $\alpha = \alpha_{M-1}$ we have

$$\|A_n g_{\alpha_{M-1}}(A_n^* A_n) A_n^* f_\delta - P_{2^{2n}} f_\delta\| > d\delta. \tag{23}$$

Owing to the inverse triangle rule it holds

$$\|Ax_{\alpha_{M-1}} - f\| \ge \|A_n g_{\alpha_{M-1}}(A_n^* A_n) A_n^* f_\delta - P_{2^{2n}} f_\delta\| - (\sqrt{2} + 1)\delta. \tag{24}$$

By spectral decomposition of the operator A we get

$$\begin{aligned}\|Ax_{\alpha_M} - f\|^2 &= \sum_{k=1}^{\infty} \lambda_k^2 \ln^{-2p} \lambda_k^{-2} (v, \psi_k)^2 \left[\frac{\lambda_k^2}{\alpha_M + \lambda_k^2} - 1\right]^2 = \\ &= \alpha_M^2 \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\left(\alpha_M + \lambda_k^2\right)^2} \ln^{-2p} \lambda_k^{-2} (v, \psi_k)^2 > \\ &> \theta^2 \alpha_{M-1}^2 \sum_{k=1}^{\infty} \frac{\lambda_k^2}{\left(\alpha_{M-1} + \lambda_k^2\right)^2} \ln^{-2p} \lambda_k^{-2} (v, \psi_k)^2.\end{aligned}$$

Hence,

$$\|Ax_{\alpha_M} - f\|^2 > \theta^2 \|Ax_{\alpha_{M-1}} - f\|^2. \tag{25}$$

Substituting (23) and (24) in (25), we finally obtain

$$\|Ax_{\alpha_M} - f\| \ge \theta(d - \sqrt{2} - 1)\delta.$$

Thus, the lemma is proved with $d_1 = \theta(d - \sqrt{2} - 1)\delta$ and $d_2 = \theta(d + \sqrt{2} + 1)\delta$.

5.2. **Error estimate of the algorithm $\mathcal{P}'$.**

Theorem 1. *Let $\|A\| \le \gamma_0 \le e^{-1/2}$, the parameters of regularization α_M and discretization n are chosen as in (20) and (19), correspondingly. Than the estimate*

$$\|x^\dagger - x_{\alpha_M,n}^\delta\| \le \tilde{c} \ln^{-p} 1/\delta \tag{26}$$

holds, where the constant $\tilde{c} > 0$ only depends on γ_0, d_1, d_2, ρ and p; $x_{\alpha_M,n}^\delta$ is determined by (9).

Proof. It is obvious that

$$\|x^\dagger - x_{\alpha_M,n}^\delta\| \le \|x^\dagger - x_{\alpha_M}\| + \|x_{\alpha_M} - x_{\alpha_M,n}\| + \|x_{\alpha_M,n} - x_{\alpha_M,n}^\delta\|.$$

Owing to 3 for the first term we have

$$\|x^\dagger - x_{\alpha_M}\| \le \xi \ln^{-p} 1/\delta.$$

By applying (13) the last term is immediately bounded

$$\|x_{\alpha_M,n} - x_{\alpha_M,n}^\delta\| = \|(\alpha_M I + A_n^* A_n)^{-1} A_n^* (f - f_\delta)\| \le \frac{\delta}{2\sqrt{\alpha_M}}.$$

Finally, we need to estimate the second term. First, consider the decomposition

$$\begin{aligned}x_{\alpha_M} - x_{\alpha_M,n} &= (\alpha_M I + A^* A)^{-1} A^* A x^\dagger - (\alpha_M I + A_n^* A_n)^{-1} A_n^* A x^\dagger = \\ &= T_1 x^\dagger + T_2 x^\dagger,\end{aligned} \tag{27}$$

where

$$T_1 := (\alpha_M I + A^*A)^{-1}A^*A - (\alpha_M I + A_n^*A_n)^{-1}A_n^*A_n,$$
$$T_2 := (\alpha_M I + A_n^*A_n)^{-1}A_n^*(A_n - A).$$

By (13), (19) and (17) we have

$$\|T_2\| \le \frac{1}{2\sqrt{\alpha_M}}\frac{4}{5\rho}\delta = \frac{2}{5\rho}\frac{\delta}{\sqrt{\alpha_M}}.$$

It is remain to estimate $\|T_1\|$. Due to (14), we rewrite T_1 as follows

$$T_1 = \alpha_M\left(\alpha_M I + A^*A\right)^{-1}\left(A^*A - A_n^*A_n\right)\left(\alpha_M I + A_n^*A_n\right)^{-1} = \overline{T}_1 + \overline{T}_2,$$

where

$$\overline{T}_1 := \alpha_M\left(\alpha_M I + A^*A\right)^{-1}A^*\left(A - A_n\right)\left(\alpha_M I + A_n^*A_n\right)^{-1},$$
$$\overline{T}_2 := \alpha_M\left(\alpha_M I + A^*A\right)^{-1}\left(A^* - A_n^*\right)A_n\left(\alpha_M I + A_n^*A_n\right)^{-1}.$$

Further, we estimate the norms of $\overline{T}_1$ and $\overline{T}_2$. Owing to (13), (19) and (17) the norm of $\overline{T}_1$ is immediately bounded as

$$\|\overline{T}_1\| \le \frac{2}{5\rho}\frac{\delta}{\sqrt{\alpha_M}}.$$

Now, we are going to estimate the norm of $\overline{T}_2$. By (14) we have

$$\overline{T}_2 = \alpha_M\left(\alpha_M I + A^*A\right)^{-1}\left(A^* - A_n^*\right)\left(\alpha_M I + A_nA_n^*\right)^{-1}A_n.$$

Applying (13), (19) and (17), we obtain

$$\|\overline{T}_2\| \le \frac{2}{5\rho}\frac{\delta}{\sqrt{\alpha_M}}.$$

Hence,

$$\|T_1\| \le \|\overline{T}_1\| + \|\overline{T}_2\| \le \frac{4}{5\rho}\frac{\delta}{\sqrt{\alpha_M}}.$$

Thus,

$$\|x_{\alpha_M} - x_{\alpha_M,n}\| \le \frac{6}{5}\frac{\delta}{\sqrt{\alpha_M}}.$$

Summing up the above bounds we finally get

$$\|x^\dagger - x^\delta_{\alpha_M,n}\| \le \xi\ln^{-p}1/\delta + \frac{6}{5}\frac{\delta}{\sqrt{\alpha_M}} + \frac{1}{2}\frac{\delta}{\sqrt{\alpha_M}} \le \xi\ln^{-p}1/\delta + \frac{17}{10}\frac{\delta}{\sqrt{\alpha_M}}.$$

Further, if α_M is chosen as in (20) and the inequality $\alpha_M \ge \delta$ holds then for sufficiently small δ we have

$$\|x_\dagger - x^\delta_{\alpha_M,n}\| \le \xi\ln^{-p}1/\delta + \frac{17}{10}\sqrt{\delta} \le \tilde{c}_1\ln^{-p}1/\delta,$$

with $\tilde{c}_1 = \xi + \frac{17}{10}$.

Otherwise, if $\alpha_M \le \delta$ then by Lemma 2 and Lemma 6 we get

$$d_1\delta \le \|Ax_{\alpha_M} - f\| \le \gamma_0^{-1}\rho\sqrt{\alpha_M}\ln^{-p}1/\alpha_M \le \gamma_0^{-1}\rho\sqrt{\alpha_M}\ln^{-p}1/\delta.$$

Thus,

$$\|x^\dagger - x^\delta_{\alpha_M,n}\| \le \xi \ln^{-p} 1/\delta + \frac{17}{10}\frac{\gamma_0^{-1}\rho}{d_1}\ln^{-p} 1/\delta = \tilde{c}_2 \ln^{-p} 1/\delta,$$

where $\tilde{c}_2 = \xi + \frac{17}{10}\frac{\gamma_0^{-1}\rho}{d_1}$. The theorem is proved with $\tilde{c} = \max\{\tilde{c}_1, \tilde{c}_2\}$. □

6. Minimal radius of Galerkin's information. Optimal order estimate

Theorem 2. *For sufficiently small δ the estimate*

$$R_{N,\delta}\left(\mathcal{H}^{r,s}_\gamma, M(A)\right) \le e_\delta\left(\mathcal{H}^{r,s}_\gamma M(A), \mathcal{P}^{‘}\right) \le c_p \ln^{-p} N^{2s}$$

is fulfilled, where $c_p > 0$ depends only on $\gamma, r, s, d_1, d_2, \rho$ and p. Moreover,

$$\mathrm{card}(\Gamma^a_n) \asymp \begin{cases} \delta^{-\frac{1}{r}}(\ln\delta^{-1})^{1+\frac{1}{2r}}, & r = s, \\ delta^{-\frac{1}{s}}, & r > s. \end{cases}$$

Proof. Rewrite the right-hand side of (26) by N, where

$$N = \begin{cases} c_1' n 2^{2n}, & r = s, \\ c_2' 2^{2an}, & r > s, \end{cases}$$

$1 < c_1' \le 2, \quad 1 + \frac{1-2^{3(1-a)}}{1-2^{1-a}} \le c_2' \le \frac{2-2^{1-a}}{1-2^{1-a}}$ (see Lemma 5). Further, we consider two cases.

First, let $r = s$. Owing to (16),(19) we have

$$\delta^{-1} = \frac{4}{5\rho\bar{c}_1} n^{-1/2} 2^{2rn} = \frac{4(c_1')^{-r}}{5\rho\bar{c}_1} N^r n^{-\frac{1}{2}-r}, \tag{28}$$

with $\bar{c}_1 = \gamma_1 2^{r+1/2}$. It is easy to see that $\ln N = \ln c_1' + 2n\ln 2 + \ln n$. It follows $n \le \frac{\ln N}{2\ln 2}$. Kipping in the mind the last inequality, from (28) we obtain the lower bound of δ^{-1}

$$\delta^{-1} \ge \frac{4(c_1')^{-r}(2\ln 2)^{1/2+r}}{5\rho\bar{c}_1} N^r (\ln N)^{-1/2-r}.$$

For any $\mu > 0$ there are some N_0 that for all $N \ge N_0$ it holds $\ln N \le N^\mu$. Hence,

$$\delta^{-1} \ge \frac{4(c_1')^{-r}(2\ln 2)^{1/2+r}}{5\rho\bar{c}_1} N^r N^{\mu(-1/2-r)} = $$
$$= \frac{4(c_1')^{-r}(2\ln 2)^{1/2+r}}{5\rho\bar{c}_2} N^{(1-\mu)r-\frac{1}{2}\mu}.$$

There are always exist μ such that $(1-\mu)r - \frac{1}{2}\mu > 0$, and the estimate (26) we can rewrite as follows

$$\|x^\dagger - x^\delta_{\alpha_M,n}\| \le c_{p,1} \ln^{-p} N^{2r}. \tag{29}$$

Now, we are going to consider the case $r > s$. Using the same arguments as above, by (16) and (19) we have

$$\delta^{-1} = \frac{4}{5\rho\bar{c}_2} 2^{2asn} = \frac{4(c_2')^{-s}}{5\rho\bar{c}_2} N^s, \tag{30}$$

where

$$\bar{c}_2 = \gamma_1 \left(1 + \frac{2^r}{1 - 2^{as-r}}\right).$$

In this case the estimate (26) we rewrite as follows

$$\|x^\dagger - x^\delta_{\alpha_M,n}\| \le c_{p,2} \ln^{-p} N^{2s}. \tag{31}$$

Taking into account the definition $R_{N,\delta}\left(\mathcal{H}^{r,s}_\gamma, M(A)\right)$, and also the relations (29) and (31) we have

$$R_{N,\delta}\left(\mathcal{H}^{r,s}_\gamma, M(A)\right) \le \|x^\dagger - x^\delta_{\alpha_M,n}\| \le c_p \ln^{-p} N^{2s},$$

where $c_p = \max\{c_{p,1}, c_{p,2}\}$.

It is remain to express the amount $\mathrm{card}(\Gamma^a_n)$ by δ. Let consider the two cases. First let $r = s$, then

$$\mathrm{card}(\Gamma^1_n) := N \asymp 2^{2n} n = (\sqrt{n} 2^{-2sn})^{-\frac{1}{s}} n^{1+\frac{1}{2s}} \asymp \delta^{-\frac{1}{s}} (\ln \delta^{-1})^{1+\frac{1}{2s}}.$$

2) Now let $r > s$, then

$$\mathrm{card}(\Gamma^a_n) := N \asymp 2^{2an} = (2^{-2asn})^{-\frac{1}{s}} \asymp \delta^{-\frac{1}{s}}.$$

Thus, summing up obtained estimates of $\mathrm{card}(\Gamma^a_n)$, we have

$$\mathrm{card}(\Gamma^a_n) \asymp \begin{cases} \delta^{-\frac{1}{r}} (\ln \delta^{-1})^{1+\frac{1}{2r}}, & r = s \\ \delta^{-\frac{1}{s}}, & r > s \end{cases}.$$

The statement of the theorem is completely proved. □

Below we formulate a result giving the order estimate of the minimal radius of the Galerkin information.

Theorem 3. *The two-side estimate*

$$\frac{1}{2^{p+1}} \ln^{-p} N^{2s} \le R_{N,\delta}\left(\mathcal{H}^{r,s}_\gamma, M(A)\right) \le c_p \ln^{-p} N^{2s}$$

holds. The indicate optimal order is achieved under the algorithm $\mathcal{P}$‘ (10), (18)–(21).

The lower bound for $R_{N,\delta}$ is established in [26], and the upper estimate was obtained in Theorem 2.

Remark 4. *Comparing results of Theorem 3 to that of [26], where the balancing principle was applied as stop rule, we can conclude that both approaches are achieved an optimal order of accuracy. Moreover, the proposed algorithm allows to provide order estimates on more wide classes of problems. Herewith, we reduce the amount of the Galerkin information (on the logarithmic multiplier) when $r = s$.*

Acknowledgment

This work was done while the authors were visiting the Johann Radon Institute for Computational and Applied Mathematics (RICAM). The authors gratefully acknowledge for the partially support in the scope of joint European Project "Approximation Methods for Molecular Modelling and Diagnosis Tools" ("AMMODIT"), Grant no: 645672.

Bibliography

1. Babenko K. I. Approximation of periodic functions of many variables by trigonometric polynomials / K. I. Babenko // Dokl. Akad. Nauk SSSR. – 1960. – Vol. 132, № 2. – P. 247-250.
2. Bauer F. A Lepskij type stopping rule for regularized Newton's method / F. Bauer, T. Hohage // Inverse Probl. – 2005. – Vol. 21, № 6. – P. 1975-1991.
3. Gfrerer H. An a posteriori parameter choice for ordinary and iterated Tikhonov regularization of ill-posed problems leading to optimal convergence rates / H. Gfrerer // Math. Comput. – 1987. – Vol. 49. – P. 507-522.
4. Hämarik U. About the balancing principal for choice of the regularization parameter / U. Hämarik, T. Raus // Num. Func. Anal. and Optim. – 2009. – Vol. 30, № 9-10. – P. 951-970.
5. Hohage T. Regularization of exponentially ill-posed problems / T. Hohage // Numer. Funct. Anal. Optim. – 2000. – Vol. 21, № 3-4. – P. 439-464.
6. Ivanov V. K. The approximate solution of operator equations of the first kind / V. K. Ivanov // Zh. Vychisl. Mat. Mat. Fiz. – 1966. – Vol. 6, № 6. – P. 1089-1094.
7. Mathe P. Discretization strategy for ill-posed problems in variable Hilbert scales / P. Mathe, S. V. Pereverzev // Inverse Problems. – 2003. – Vol. 19, № 6. – P. 1263-1277.
8. Morozov V. A. Regularization of incorrectly posed problems and the choice of regularization parameter / V. A. Morozov // Zh. Vychisl. Mat. Mat. Fiz. – 1966. – Vol. 6, № 1. – P. 170-175.
9. Morozov V. A. Choice of parameter in solving functional equations by the method of regularization. / V. A. Morozov // Dokl. Akad. Nauk SSSR. – 1967. – Vol. 175. – P. 1225–1228.
10. Pereverzev S. V. Hyperbolic cross and complexity of an approximate solution of Fredholm integral equations of the second kind with differentiable kernels / S. V. Pereverzev // Sibirsk. Mat. Zh. – 1991. – Vol. 32, № 1. – P. 107-115.
11. Pereverzev S. V. On the complexity of the problem of finding the solutions of Fredholm equations of the second kind with smooth kernels. I / S. V. Pereverzev // Ukrain. Mat. Zh. – 1988. – Vol. 40, № 1. – P. 84-91.
12. Pereverzev S. V. On the complexity of the problem of finding the solutions of Fredholm equations of the second kind with smooth kernels. II. / S. V. Pereverzev // Ukrain. Mat. Zh. – 1989. – Vol. 41, № 2. – P. 189-193.
13. Pereverzev S. Information Complexity of Equations of Second Kind with Compact Operators in Hilbert Space / S. Pereverzev, C. Scharipov // J. Complexity. – 1992. – Vol. 8. – P. 176-202.
14. Pereverzev S. On the adaptive selection of the parameter in regularization of ill-posed problems / S. Pereverzev, E. Schock // SIAM J. Numer. Anal. – 2005. – Vol. 43, № 5. – P. 2060-2076.
15. Pereverzev S. V. Optimization of Projection Methods for solving Ill-Posed Problems / S. V. Pereverzev // Computing. – 1995. – Vol. 55. – P. 113-124.
16. Pereverzev S. V. The minimal radius of Galerkin information for the Fredholm problems of first kind / S. V. Pereverzev, S. G. Solodky // J. Complexity. – 1996.– Vol. 12, № 4. – P. 401-415.
17. Plato R. The Lavrentiev-regularized Galerkin method for linear accretive ill-posed problems / R. Plato // Matimyás Mat. – 1998. – Vol. 21. – Special Issue. – P. 57–66.

18. Plato R. On the regularization of projection methods for solving ill-posed problems / R. Plato, G. M. Vainikko // Numer. Math. – 1990. – Vol. 57. – P. 63-79.
19. Raus T. Residue principle for ill-posed problems / T. Raus // Acta et comment. Univers. Tartuensis. – 1984. – Vol. 672. – P. 16-26.
20. Schock E. Morozov's discrepancy principle for Tikhonov regularization of severely ill-posed problems in finite-dimensional subspaces / E. Schock, S. V. Pereverzev // Numer. Funct. Anal. Optim. – 2000. – Vol. 21, № 7-8. – P. 901-916.
21. Solodky S. G. Optimal approximation for solving linear ill-posed problems / S. G. Solodky // J.Complexity. – 1999.– Vol. 15. – P. 123-132.
22. Solodky S. G. Optimization of the projection methods for solving linear ill-posed problems / S. G. Solodky // Zh. Vychisl. Mat. Mat. Fiz. – 1999.– Vol. 39, № 2. – P. 195-203.
23. Solodky S. G. On optimization of projection methods for solving some classes of severely ill-posed problems / S. G. Solodky, G. L. Myleiko // J. Appl. Anal. – 2016.– Vol. 95, № 4.– P. 826-841.
24. Solodky S. G. The minimal radius of Galerkin information for severely ill-posed problems / S. G. Solodky, G. L. Myleiko // Journal of Inverse and Ill-Posed Problems. – 2014. – Vol. 22, № 5. – P. 739-757.
25. Solodky S. G. About Regularization of Severely Ill-Posed Problems by Standard Tikhonov's Method with the Balancing Principle / S. G. Solodky, G. L. Myleiko // Mathematic Modelling and Analysis. – 2014. – Vol. 19, № 2. – P. 199-215.
26. Solodky S. G. About minimal informational efforts by solving exponentially ill-posed problems / S. G. Solodky, E. V. Semenova // Journal of Num. and Appl. Math. – 2015. – Vol. 2. – P. 90-100.
27. Tautenhahn U. The use of monotonicity for choosing the regularization parameter in ill-posed problems / U. Tautenhahn, U. Hämarik // Inverse Problems. – 1999. – Vol. 15. – P. 1487-1505.
28. Traub J. F. Information-Based Complexity / J. F. Traub, G. Wasilkowski, H. Wazniakowski. – Boston: Academic Press, 1988. – P. 523.
29. Traub J. F. A general theory of optimal algorithms. / J. F. Traub, H. Wazniakowski. – Boston: Academic Press, 1980. – 382 p.
30. Vainikko G. M. Iteration procedures in ill-posed problems / G. M. Vainikko, A. Yu. Veretennikov. – Nauka, Moscow, 1986. – 182 p.

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Received 11.05.2016