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ANOTHER CASE OF INCIDENCE MATRIX FOR BIVARIATE BIRKHOFF INTERPOLATION

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Резюме. У цій статті спершу подано спеціальний випадок одновимірної задачі інтерполяції Біркгофа і за її допомогою апроксимовано розв'язок граничної задачі для рівняння Лапласа. Далі розглянуто інший тип двовимірної задачі інтерполяції Біркгофа, в якій умови інтерполяції задані в точках з кратністю. Введено інше позначення для матриці інцидентності. Зроблено порівняння апроксимацій Біркгофа і Хаара і показано перевагу інтерполяції Біркгофа.

Abstract. In this paper, first we present a special case of the univariate Birkhoff interpolation problem, and using that, we approximate the solution of a Laplace boundary value problem. Then we present another type of bivariate Birkhoff interpolation problem in which interpolation conditions are on some knots with multiplicity. We introduce another notation for incidence matrix. Finally, we compare two approximations Birkhoff and Haar then we show that Birkhoff interpolation is better than the other.

1. Introduction

In this paper we present some basic notations and useful properties in analyzing the interpolation polynomials. Let us denote Π_n the space of one variable interpolation polynomials of degree not exceeding n, and Π_n^2 the space of bivariate interpolation polynomials of degree not exceeding n.

The problem of interpolating a real function f by a univariate polynomial from the values of f and some of its derivatives on a set of knots is one of the main questions in numerical analysis and approximation theory.

In [1] and [10] the authors studied univariate Birkhoff interpolation and its properties. Let $x = \{x_1, ..., x_n\}$ be a set of real numbers such that $x_1 < ... < x_n$, let r be an integer and let $I \subset \{1, ..., n\} \times \{0, .., r\}$ be the set of pairs (i, j) in which the value $f^{(j)}(x_i) = f_{i,j}$ is known where f is a real function. The problem of determining the existence and uniqueness of a polynomial P in $\mathbb{R}^1$ satisfying the conditions $\forall (i, j) \in I, p^{(i)}(x_i) = f_{i,j}$ is called the *Birkhoff interpolation problem.*

In recent years there has been renewed interest and progress on Hermite-Birkhoff interpolation. The original source for this activity is work by G. D. Birkhoff in 1906, with a notable contribution by G. Polya in 1931.

Key words. Bivariate Birkhoff Interpolation Problem; Polya Condition; Incidence Matrix; Interpolation Polynomial; Haar Approximation; Hermite-Birkhoff; Operator Interpolation.

The interpolation conditions can be described by using special type matrices. Consider the matrix $E = (e_{i,j})$ with n rows and r+1 columns, filled with 0's and 1's so that $e_{i,j} = 1$ if and only if $(i, j) \in I$. The E is called *incidence matrix.*

In 1966 Schoenberg (see [19]) posed the problem of determining all those E for which the problem $P^{(j)}(x_i i) = c_{i,j}$ is always (for all choice of $x_i, c_{i,j}$) solvable. We call such matrices E *regular* and the remaining matrices *singular*.

Let $E = (e_{i,j})$ be an $m \times (n+1)$ incidence matrix. Then $m_j = \sum_i e_{i,j}$ is the number of 1's in column j, and $M_r = \sum_{j=0}^{r} m_j = \sum_{j=0}^{r}\sum_{i=1}^{m} e_{i,j}$ is the number of 1's in columns of E numbered $0, 1, \ldots, r$. For the matrix E, the condition $M_r \geq r+1, r = 0.1, ..., n$, is called the Polya condition.

Definition 7. The incidence matrix $E = (e_{i,j}), 1 \leq i \leq m, 0 \leq j \leq n$ is called poised with respect to $\{x_i\}_{i=1}^{m}$ if the unique solution of problem $P^{(j)}(x_i) = 0, 1 \leq i \leq m, 0 \leq j \leq n$ is a trivial polynomial.

In [8], the following Polya's result is well-known.

Theorem 1 (Polya's Theorem)**.** *The incidence matrix E of* $2 \times n$ *dimension is poised if and only if Polya condition is true.*

In [20], the author posed, for a $2 \times n$ incidence matrix $E = (e_{i,j})$, we define a $2 \times n$ matrix $G = (g_{i,j})$ as follows:

$$g_{i,j} = 1 - e_{i,n-j-1}, 1 \leq i \leq 2, 0 \leq j \leq n-1.$$

Then G is also an incidence matrix, because $\sum_{i=1}^{2}\sum_{j=0}^{n-1} e_{i,j} = n$. The matrix G is called a dual incidence matrix corresponding to E. For example, for the incidence matrix $E = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{Vmatrix}$, its dual matrix becomes $E' = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$.

The following theorem give a relationship between a $2 \times n$ incidence matrix E and its dual matrix G.

Theorem 2. *A* $2 \times n$ *incidence matrix E is poised if and only if dual matrix G is poised. In [20], the author shown that there exists a quadrature formula in the form* $\int_a^b P(x)dx = \sum_{e_{i,j}=1} w_{i,j} P^{(j)}(x_i)$ *to be exact for any polynomial P with degree at most n-1, where* $w_{i,j}$*'s are weight coefficients independent of P. This is called, the Hermite-Birkhoff quadrature formula for the incidence matrix E.*

Theorem 3. *A* $2 \times n$ *incidence matrix E is poised if and only if there exists a Hermite-Birkhoff quadrature formula specified by E.*

In [16], author presented below property of incidence matrix E:

Let $m_j = \sum_i e_{i,j}, j = 0, ..., n$ and $M_r = \sum_{j=0}^{r} m_j, r = 0, ..., n$, then E satisfies the strong Polya condition if $M_r \geq r+2, r = 0, ..., n-1$. If E does not satisfy strong Polya condition, then E may be decomposed in to matrices, $E = E_1 \bigoplus E_2 \bigoplus \cdots \bigoplus E_N$ where E_j's satisfies strong Polya condition.

In [18], the author proved below theorem:

Theorem 4. *Let* E_n *satisfy the Polya condition. Then* E_n *has a unique decomposition* $E_n = E_{n_1} \bigoplus E_{n_2} \bigoplus ... \bigoplus E_{n_q}, n_1+n_2...+n_q = n$*, where* $E_{n_j}, j = 1, ..., q$

satisfies the strong Polya condition. Moreover E_n is poised if and only if E_{n_j}'s are poised.

The Birkhoff interpolation problem is one of the most general problems in multivariate interpolations. For clarity of the exposition, we will only restrict ourselves to the bivariate case.

In [11, Def. 3.1.1, p. 9], the authors studied bivariate Birkhoff interpolation problem. The bivariate Birkhoff interpolation problem depends on a finite set $T = \{z_q\}_{q=1}^{m} \subset \mathbb{R}^2$ of knots and interpolation space Π_n^2 of polynomials and an incidence matrix $E = (e_{q,\alpha})$. The bivariate Birkhoff interpolation problem is, for given real numbers $c_{q,\alpha}$, to find a polynomial $p \in \Pi_n^2$ satisfying the interpolation conditions

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial y^{\alpha_2}\partial x^{\alpha_1}} p(z_q) = c_{q,\alpha} \tag{1}$$

with $e_{q,\alpha} = 1$ where $\alpha = (\alpha_1, \alpha_2)$.

In this paper, we present a special case of univariate Birkhoff interpolation problem together with an example of boundary value problem introduce in [2], and also a method for obtaining the interpolation polynomial in the case of a set of types conditions, given on a set of knots in $\mathbb{R}^2$. This method is a generalization of the tensorial product method introduced by F.J.Hack in [7]. In this way, we investigate bivariate Birkhoff polynomial for the set of knots T such that $|T| < \binom{n+2}{2}$.

In [11] and [3], authors introduced Polya conditions for multivariate Birkhoff interpolation as follows:

Definition 8. An incidence matrix E satisfies the (lower) Polya condition (with respect to S) if $|E_A| \le |A|$ for any lower set $A \subseteq S$. E satisfies the upper Polya condition if $|E_B| \le |B|$ for any upper set $B \subseteq S$. A set B is an upper set with respect to S if $\alpha \in B, \beta \ge \alpha$ and $\beta \in S$ imply that $\beta \in B$. B is an upper set of S if and only if $S \setminus B$ is a lower set.

Similar to notations in [4], we apply the Haar function and interpolation problem. In [4], the authors presented some theorems for uniqueness. Thus we employ those theorems, for example, formula (8) and Theorem 3.1 and Example 3.2, p.107-109.

Problems of generalization in functions interpolation theory with functionals and operators in abstract spaces are considered in numerous works.

Definition 9. Let $F : X \longrightarrow Y$ be an operator, where X is a Hilbert and Y is a vector space; let $P_n : X \longrightarrow Y$ be an operator polynomial of the form $P_n(x) = L_0 + L_1x + ... + L_nx^n$, where $L_0 \in Y$; and let $L_p(t_1, ..., t_p) : X^p \longrightarrow Y$ be a p-linear operator, $p = 1, ..., n$. Let $\{x_i\}_{i=1}^{m}$ be a system of elements from X. A polynomial operator P_n is called an interpolating polynomial for F in nodes $\{x_i\}_{i=1}^{m} \subset X$ if it satisfies the conditions $P_n(x_i) = F(x_i), i = 1, ..., m$.

In the case $X = Y = R^1$ the requirement that the interpolation functionals be the same algebraic polynomials.

In [12] and [13], the authors investigated the operator interpolation theory in Hilbert space and solvability Hermite interpolation problem with the operator values at the nodes with Gateaux differentials defined on the auxiliary nodes and some given directions. For example, let Π_n be a set of the operator polynomials $P_n : X \longrightarrow Y$ of degree not exceeding n and $p \in \Pi_n$ satisfies the conditions:

$$p(x_i) = F(x_i), p'(x_i)h_i = F'(x_i)h_i, i = 1, ..., m \tag{2}$$

For investigate Hermite problem with interpolation conditions (2) we consider the auxiliary nodes

$$\overline{x}_1 = x_1, \overline{x}_2 = x_1 + \alpha h_1, \overline{x}_3 = x_2, \overline{x}_4 = x_2 + \alpha h_2, ..., \overline{x}_{2m-1} = x_m,$$

$$\overline{x}_{2m} = x_m + \alpha h_m, \alpha \in R^1, \alpha \neq 0$$

of the matrix

$$\Gamma(\alpha) = \left\| \textstyle\sum_{p=0}^{n} (\overline{x}_i, \overline{x}_j)^p \right\|_{i,j=1}^{2m}$$

and

$$C(\alpha) = \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ \frac{-1}{\alpha} & \frac{1}{\alpha} & 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{\alpha} & \frac{1}{\alpha} & 0 & ... & 0 & 0 & 0 \\ . & . & . & . & . & ... & . & . & . \\ . & . & . & . & . & ... & . & . & . \\ . & . & . & . & . & ... & . & . & . \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & ... & 0 & \frac{-1}{\alpha} & \frac{1}{\alpha} \end{matrix} \right\|$$

and the vectors

$$F(\alpha) = (F(\overline{x}_1), F(\overline{x}_2), ..., F(\overline{x}_{2m})), P(\alpha) = (p(\overline{x}_1), p(\overline{x}_2), ..., p(\overline{x}_{2m})), p \in \Pi_n.$$

In [13], p.97, Theorem 1.1 shown that a necessary and sufficient condition for the solvability of the Hermite operator interpolation problem (2) in a Hilbert space, that the condition $ZF_H = 0$ and the formula $p(x) = q(x) + \langle F_H - q_H, H^+ g_H(x) \rangle$, with $q(x)$ varies over Π_n, describes the whole set of the Hermite operator polynomials of the n-th degree satisfies the interpolation conditions (2). In [13], explained notations $ZF_H, F_H, q_H, H^+ g_H$.

When some of the conditions of the Hermite interpolation are absent then, they are called to Hermite- Birkhoff conditions. For example, the conditions:

$$p(x_i) = F(x_i), p''(x_i)h_2^{(2)}h_1^{(2)} = F''(x_i)h_2^{(2)}h_1^{(2)}, i = 1, ..., m \tag{3}$$

are Hermite-Birkhoff conditions. In [13], Theorem 2.1, p.110, itroduced a necessary and sufficient condition for the solvability of the Hermite-Birkhoff operator interpolation problem in a Hilbert space.

Now we introduce an important result as follows:

A sufficient condition for the invariant solvability of the Hermite operator interpolation problem given as the following theorem. We shall denote by M a number of the interpolation conditions in the Hermite operator interpolation.

Theorem 5. *The Hermite interpolation problem in a Hilbert space is invariant solvability for any* $n \geq M - 1$.

For example every Hermite interpolation problem with conditions (2) by Theorem 5 is invariant solvable.

By text in [13], p.112, if the Hermite-Birkhoff interpolation problem for a function of one variable has the unique solution, then the appropriate Hermite-Birkhoff operator interpolation problem is invariantly solvable. Now we apply Polya's theorem in case m=2 for the invariant solvability of the Hermite-Birkhoff operator problem with the interpolation conditions containing values of operator polynomial p of the third degree and Gateaux differentials of the second order

$$p(x_1), p''(x_1)h_2^{(1)}h_1^{(1)}, p(x_2), p''(x_2)h_2^{(2)}h_1^{(2)} \tag{4}$$

In the corresponding Hermite-Birkhoff interpolation problem of one variable we have

$$M = 4, \quad n = M - 1 = 3, \quad E = \left\| \begin{matrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{matrix} \right\|,$$

$$m_0 = 2, \quad m_1 = 0, \quad m_2 = 2, \quad M_0 = 2, \quad M_1 = 2, \quad M_2 = 4$$

Since $M_j \geq j + 1, j = 0, 1, 2$ then by Polya's Theorem, the classical Hermite-Birkhoff problem

$$p(t_1) = 0, \quad p''(t_1) = 0, \quad p(t_2) = 0, \quad p''(t_2) = 0$$

on the set of the polynomial of the 3-d degree has the unique solution zero-polynomial. But as we stated above, the corresponding Hermite-Birkhoff operator problem (4) is invariantly solvable.

2. Bivariate Birkhoff Interpolation

Following R.A. Lorentz in [11], an interpolation problem is *regular* if it is uniquely solvable for all selections of distinct nodes and all data. In the univariate case, Lagrange and Hermite interpolation are regular, but in the multivariate case, even Lagrange interpolation is not regular. Here, we study a solvable interpolation problem in multivariate case.

A uniqueness technique for bivariate Birkhoff interpolation problem is presented in [7]. The technique has been explained in [7, theorem 3.3, p.26], where interpolation polynomial is tensor product of functionals. that is why, we introduce incidence matrix. For exactly M+1 pairs $(i, k) \in \{1, ..., m\} \times \{0, ..., M\}$, we suppose that $E_{i,k} = (e_{i,j}^{k,l})_{1 \leq j \leq a_{i,k}, 0 \leq l \leq N_{i,k}}$ where $a_{i,k} \in N, N_{i,k} \in N_0$ and for others $(i, k)'s, E_{i,k} = 0$. Regularity condition is established, using bidimensional incidence matrix corresponding to Birkhoff interpolation problem. Hence, the bivariate Birkhoff interpolation problem is as follows:

$$C^M(G), \sum_{s=1}^{p} \Pi_{M_s} \otimes \Pi_{N_s}; D_{x_i, y_{i,k,j}}^{k,l} : (i, k) \in Z, (x_i, y_{i,k,j}) \in T \tag{5}$$

This means that for all $f \in C^M(G), G \subset \mathbb{R}^2$ there exists $P \in \sum_{s=1}^{p} \Pi_{M_s} \otimes \Pi_{N_s}$ where $\Pi_{M_s} \otimes \Pi_{N_s}$ is tensor product of functionals and $D_{x_i, y_{i,k,j}}^{k,l} P = D_{x_i, y_{i,k,j}}^{k,l} f$ such that T is the set of distinct knots i.e. $T = \{(x_i, y_{i,k,j})\}$ s.t. $x_1 < ... < x_m$

and also $y_{i,k,1} < ... < y_{i,k,a_{i,k}}, (i,k) \in Z, Z \subset \{1,...,m\} \times \{0,...,M\}$ so that $Z = \{(i,k) : E_{i,k} \neq 0\}$.

In view of corollary 3.4 [7, p.27], if matrices E_s's for points $x_1,...,x_m$ are regular and matrices $E_{i,k}$'s are regular for points $y_{i,k,1},...,y_{i,k,a_{i,k}}$ then the incidence matrix $\varepsilon_{m,M}$ is regular for $\{(x_i, y_{i,k,j})\}$. It means that the interpolation problem is unique.

3. The Result

3.1. **Univariate Case.** In [2], a Birkhoff interpolation problem was studied. Now, we introduce another case of Birkhoff interpolation problem. In [17], the author introduced Lagrange's fundamental polynomials. For given points $x_0, x_1, ..., x_n$, let us use the fundamental polynomials $l_0, l_1, ..., l_n$, where $l_i(x) = \prod_{j\neq i}(\frac{x - x_j}{x_i - x_j})$ such that

$$l_i(x_k) = \begin{cases} 1 & \text{if} k = i \\ 0 & \text{if} k \neq i \end{cases}, \qquad k, i = 0, 1, ..., n.$$

We recall that the Green's function was defined in [5], [6], [14].

Theorem 6. *Let $\omega_i \in \mathbb{R}^1, i = 0, 1, ..., n$ and $-1 = x_0 < x_1 < ... < x_{n-1} < x_n = 1$ and $l_i(x)$ be the fundamental polynomials of Lagrange calculated on the n-1 points $x_i, i = 1, ..., n-1$ and $p_{n,i}(x) = \int_{-1}^{1} G(x,t) l_i(t) dt, i = 1, ..., n-1$, where*

$$G(x,t) = \begin{cases} 1 & t < x \\ 0 & x < t \end{cases}$$

is a Green's function, then the polynomial

$$P_n(x) = \begin{cases} \omega_n & x = x_n \\ \omega_0 + \sum_{i=1}^{n-1} p_{n,i}(x)\omega_i & \text{otherwise} \end{cases} \tag{6}$$

is the unique polynomial of degree $\leq n-1$ which satisfies the Birkhoff interpolation conditions

$$P_n(x_0) = \omega_0,\ P'_n(x_i) = \omega_i,\ i = 1, ..., n-1,\ P_n(x_n) = \omega_n \tag{7}$$

Proof. We know that $P_{n,i}(x)$ is the solution of the boundary value problem

$$\begin{cases} P'_{n,i}(x) = l_i(x) \\ P_{n,i}(-1) = 0 \end{cases}, \quad i = 1, ..., n-1, \quad \text{because} \quad P_{n,i}(x) = \int_{-1}^{x} l_i(t) dt.$$

The polynomial (6) satisfies the interpolatory conditions (7). For the proof of the uniqueness, since $P_{n,i}(x)$ is a polynomial of degree not exceeding n-1, now suppose that $\overline{P}_n$ is another polynomial of degree not exceeding n-1 where it is true in (7) such that $\overline{P}_n(x) \neq P_n(x)$.

We set $\phi_n(x) := \overline{P}_n(x) - P_n(x)$. The polynomial $\phi_n(x)$ has n-1 zeros, therefore it has n-2 optimum, namely, $\phi'_n(x_i) = \overline{P}'_n(x_i) - P'_n(x_i) = 0$. After repeated this process and applying Rolle's theorem, we conclude that $\phi_n(x) \equiv 0$. Thus $\overline{P}_n(x) = P_n(x)$ that is contradiction. □

Remark 1. *By Theorem 6, since the Hermite-Birkhoff interpolation problem with conditions (7) has unique solution then the corresponding Hermite-Birkhoff operator interpolation problem is invariant solvable.*

In [9] and [15], the authors presented Haar approximation. Now, we introduce Haar function and its apply for below example.

Definition 10. The Haar function $\chi_n(x), x \in [0,1]$, where $\chi_1 \equiv 1$, and for $2^k < n \leq 2^{k+1}, k = 0, 1, \cdots$ is defined as follows:

$$\chi_n(x) = \begin{cases} 2^{\frac{k}{2}} & \text{x} \in \Delta_n^+ \\ -2^{\frac{k}{2}} & \text{x} \in \Delta_n^- \\ 0 & \text{x} \notin \overline{\Delta_n} \end{cases} \tag{8}$$

where Δ_n is a binary interval of the form $(\frac{i-1}{2^k}, \frac{i}{2^k})$ where $k = 0, 1, ...$ and $i = 1, 2, ..., 2^k$. For $n = 2^k + i$ we write $\Delta_n = \Delta_k^i = (\frac{i-1}{2^k}, \frac{i}{2^k}), \overline{\Delta_n} = [\frac{i-1}{2^k}, \frac{i}{2^k}], \Delta_1 := \Delta_0^0 = (0,1), \overline{\Delta_1} = [0,1], \Delta_n^+ = (\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}), \Delta_n^- = (\frac{2i-1}{2^{k+1}}, \frac{i}{2^k})$ The values of $\chi_n(x)$ at points of discontinuity and at the endpoints of $[0,1]$ are specified as follows: $\chi_n(x) = \frac{1}{2} lim_{a \to 0}[\chi_n(x+a) + \chi_n(x-a)], x \in (0,1), \chi_n(0) = lim_{t \to 0^+} \chi_n(t)$, $\chi_n(1) = lim_{t \to 0^+} \chi_n(1-t)$.

For clarity of the Theorem 6, we present an example:

Example 1. The solution of Laplace boundary value problem

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0 < x < 1 \, , 0 < y < 1 \\ u(0,y) = u(1,y) = 0 \\ u(x,0) = 0 \\ u(x,1) = e^x \end{cases}$$

is $u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi y) \sin(n\pi x)$ where $b_n = \frac{2n\pi - 2ne\pi(-1)^n}{(1+n^2\pi^2) sinh(n\pi)}$.

Using theorem 6, we compute Birkhoff interpolation polynomial for $f(x) = e^x$ in these knots: $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$. Let $\omega_0 = f(0) = 1, \omega_1 = f'(\frac{1}{4}) = e^{\frac{1}{4}}, \omega_2 = f'(\frac{1}{2}) = e^{\frac{1}{2}}, \omega_3 = f'(\frac{3}{4}) = e^{\frac{3}{4}}, \omega_4 = f(1) = e$ then, $l_1(t) = 8t^2 - 10t + 3, l_2(t) = -16t^2 + 16t - 3, l_3(t) = 8t^2 - 6t + 1$ are Lagrange polynomials on x_1, x_2, x_3 and $P_{4,1}(x) = \frac{8}{3}x^3 - 5x^2 + 3x, P_{4,2}(x) = \frac{-16}{3}x^3 + 8x^2 + 3x, P_{4,3}(x) = \frac{8}{3}x^3 - 3x^2 + x$ thus $P_4(x) = (\frac{8}{3}e^{\frac{1}{4}} - \frac{16}{3}e^{\frac{1}{2}} + \frac{8}{3}e^{\frac{3}{4}})x^3 + (-5e^{\frac{1}{4}} + 8e^{\frac{1}{2}} - 3e^{\frac{3}{4}})x^2 + (3e^{\frac{1}{4}} - 3e^{\frac{1}{2}} + e^{\frac{3}{4}})x + 1$ and also $p_4(1) = e$.

Now, we employ approximation Haar-Fourier $P_H(x) = \sum_{n=1}^{\infty} C_n(f)\chi_n(x)$ for the function $f(x) = e^x$ and its compare to $P_4(x)$. First, we compute Haar-Fourier coefficients $C_n(f) = \int_0^1 f(x)\chi_n(x)dx$ as follows: $C_1(f) = e - 1, C_2(f) = 2e^{1/2} - e - 1, C_3(f) = 2\sqrt{2}e^{1/4} - \sqrt{2}e^{1/2} - \sqrt{2}, C_4(f) = 2\sqrt{2}e^{3/4} - \sqrt{2}e^{1/2} - \sqrt{2}e$, thus the Haar polynomial is: $P_H(x) = e - 1 + (2e^{1/2} - e - 1)\chi_2(x) + (2\sqrt{2}e^{1/4} - \sqrt{2}e^{1/2} - \sqrt{2})\chi_3(x) + (2\sqrt{2}e^{3/4} - \sqrt{2}e^{1/2} - \sqrt{2}e)\chi_4(x)$ Using the following ten points, we compare $f(x), P_4(x), P_H(x)$

TABL. 1. Comparison of $f(x)$, $p_4(x)$, $p_H(x)$ in $[0, 1]$

x	$f(x)$	$p_4(x)$	$p_H(x)$
0	1	1	1.136101666
0.1	1.105170918	1.106753896	1.136101666
0.25	1.284025417	1.286209257	1.297442541
0.3	1.349858808	1.352009578	1.458783416
0.5	1.648721271	1.650644616	1.665949200
0.7	2.013752707	2.020817622	1.873114984
0.75	2.117000017	2.119201800	2.139121116
0.9	2.459603111	2.461087206	2.405127248
0.99	2.691234472	2.691012369	2.405127248
1	2.718281828	2.717776531	2.405127248

Consequently one might favor Birkhoff interpolation in some cases.
Now, we set $p_4(x)$ instead of $f(x) = e^x$ in Laplace boundary value problem and obtain the approximation solution $u(x, y) = \Sigma_{n=1}^{\infty} a_n \sinh(n\pi y) \sin(n\pi x)$ where $a_n = \frac{2}{\sinh(n\pi)} \int_0^1 p_4(x) \sin(n\pi x) dx$.

Using Maple program, graphs are as follows:

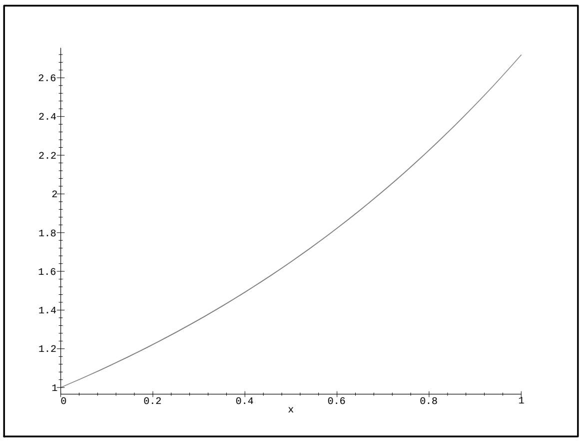

FIG. 1. Comparison of $f(x)$ with $p_4(x)$ in [0,1]

3.2. **Bivariate Case.** In this paper, uniqueness is investigated in another way. In [7, Corollary 3.4, p.27], if the incidence matrix is characterized, then the interpolation polynomial can be obtained.

Now, suppose that the interpolation conditions are given. Then, we compute a corresponding matrix as follows:

Theorem 7. *Suppose that the bivariate Birkhoff interpolation problem (5) is given. Then, the incidence matrix is characterized.*

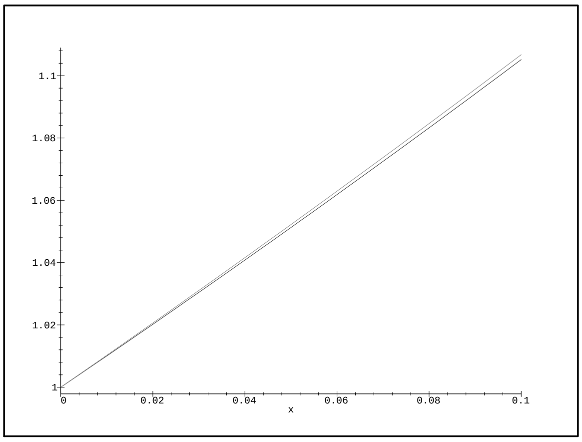

FIG. 2. Comparison of $f(x)$ with $p_4(x)$ in [0,0.01]

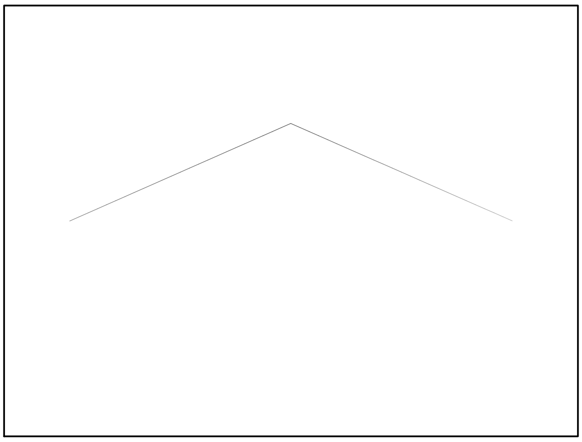

FIG. 3. Graph $u\,(x,y) = \sum_{n=1}^{1000} b_n \sinh{(n\pi y)} \sin{(n\pi x)}$

Proof. First, we arrange the knots as follows $x_1 < ... < x_m$, where for each the second component of these points, namely, $y_{i,k,1}, ..., y_{i.k.j}$, $1 \leq j \leq a_{i,k}$, where $a_{i,k} \in \mathbb{N}$.

Note that k is the order of partial derivative of the first variable for P(x,y), and we denote the order of partial derivative of second variable for P(x,y) by l, where $0 \leq l \leq N_{i,k}$, $N_{i,k} \in \mathbb{N}_0$.

Let Z be a set of pairwise (i,k)'s in (5). For indices i,j,k,l in (5), we define $e_{i,j}^{k,l} = 1, (i,k) \in Z$.

Using $e_{i,j}^{k,l}$, we construct a matrix where j,l are the number of rows and columns, respectively.

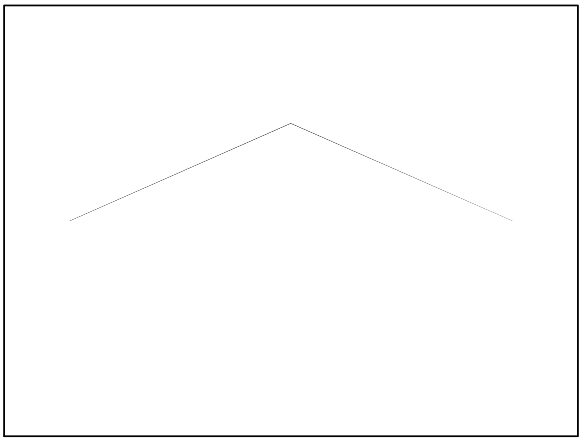

FIG. 4. Graph $u(x,y) = \sum_{n=1}^{1000} a_n \sinh(n\pi y) \sin(n\pi x)$

Let $E_{i,k} = (e_{i,j}^{k,l})_{j=1,l=0}^{a_{i,k}\ N_{i,k}}, (i,k) \in Z$. Regarding $E_{i,k}$, the number of rows and columns are equal $a_{i,k}$ and $N_{i,k} + 1$ respectively. It means that for every $(i,k) \in Z$ the value of $e_{i,j}^{k,l}$ is equal 1 otherwise is equal 0. But for the other points $(i,k) \in \{1,...,m\} \times \{0,...,M\}$ every array of $E_{i,k}$ equals zero where

$$M = |Z| - 1 \tag{9}$$

Thus, for the bivariate Birkhoff interpolation problem (5), the corresponding matrix is $\varepsilon_{m,M} = (E_{i,k})_{i=1,k=0}^{m\ \ M}$ that is an incidence matrix. □

Now, we present two examples as follows and apply Theorem 7 to obtain interpolation polynomial. In the first example, we use incidence matrix and obtain interpolation polynomial. In the second example, we use interpolation conditions and obtain interpolation polynomial.

Example 2. Consider bivariate incidence matrix

$$\varepsilon_{4,4} = \left\| \begin{matrix} 0 & \left\| \begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{matrix} \right\| & 0 & 0 & 0 \\ \left\| \begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix} \right\| & 0 & 0 & 0 & 0 \\ 0 & \left\| \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\| & \left\| \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \right\| & 0 & 0 \\ \left\| \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \right\| & 0 & 0 & 0 & 0 \end{matrix} \right\| \tag{10}$$

In view of Theorem 7, we have

$$Z = \{(1,1), (2,0), (3,1), (3,2), (4,0)\}, \quad m = M = 4,$$

$$E_{1,1} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix} \Rightarrow a_{1,1} = 3, \quad N_{1,1} = 3,$$

$$E_{2,0} = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{Vmatrix} \Rightarrow a_{2,0} = 3, \quad N_{2,0} = 4,$$

$$E_{3,1} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \Rightarrow a_{3,1} = 4, \quad N_{3,1} = 3,$$

$$E_{3,2} = \begin{Vmatrix} 1 & 0 \\ 1 & 0 \end{Vmatrix} \Rightarrow a_{3,2} = 2, \quad N_{3,2} = 1,$$

$$E_{4,0} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \Rightarrow a_{4,0} = 2, \quad N_{4,0} = 14$$

Using (3.6) in [7], we have
$N_1 = N_{2,0} = 4$, $M_1 = 0$
$N_2 = N_{1,1} = N_{3,1} = 3$, $M_2 = 2$
$N_3 = N_{3,2} = N_{4,0} = 1$, $M_3 = 4$
Consider the following points in $[0,1]^2$

$$\begin{cases} x_1 = 0, x_2 = 0.1, x_3 = 0.9, x_4 = 1 \\ y_{1,1,1} = 0, y_{1,1,2} = 0.1, y_{1,1,3} = 0.2 \\ y_{2,0,1} = 0, y_{2,0,2} = 0.2, y_{2,0,3} = 0.5 \\ y_{3,1,1} = 0, y_{3,1,2} = 0.3, y_{3,1,3} = 0.6, y_{3,1,4} = 0.9 \\ y_{3,2,1} = 0.8, y_{3,2,2} = 1 \\ y_{4,0,1} = 0, y_{4,0,2} = 1 \end{cases} \tag{11}$$

Since the incidence matrices $E_{i,k}$'s are regular and

$$E_1 = \begin{Vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Vmatrix}, \quad E_2 = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad E_3 = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

are also regular, then by [7, Corollary 3.4,p.27], the incidence matrix $\varepsilon_{4,4}$ is regular. So, bivariate Birkhoff interpolation problem
$(C^q([0,1]^2), \Pi_0 \otimes \Pi_4 + \Pi_2 \otimes \Pi_3 + \Pi_4 \otimes \Pi_1; D^{k,l}_{x_i, y_{i,k,j}} : (i,k) \in Z, (x_i, y_{i,k,j}) \in T)$,
where
$q = max\{M_s + N_s\}_{s=1}^{3} = 5$, and $x_1 < ... < x_4, y_{i,k,1} < ... < y_{i,k,a_{i,k}}$
is uniquely solvable.

That is for all $f \in C^5([0,1]^2)$ for example $f(x,y) = ye^x$ there exists

$$P \in \sum_{s=1}^{3} \Pi_{M_s} \otimes \Pi_{N_s}$$

i.e.
$P(x,y) = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy +$

$+a_{0,2}y^2 + a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3 + a_{4,0}x^4 +$
$+a_{3,1}x^3y + a_{2,2}x^2y^2 + a_{1,3}xy^3 + a_{0,4}y^4 + a_{4,1}x^4y + a_{2,3}x^2y^3$
and

$$\begin{cases} \frac{\partial^3 P}{\partial y^2 \partial x}(0,0) = \frac{\partial^3 f}{\partial y^2 \partial x}(0,0) \\ \frac{\partial P}{\partial x}(0,0.1) = \frac{\partial f}{\partial x}(0,0.1) \\ \frac{\partial^3 P}{\partial y^2 \partial x}(0,0.1) = \frac{\partial^3 f}{\partial y^2 \partial x}(0,0.1) \\ \frac{\partial^2 P}{\partial y \partial x}(0,0.2) = \frac{\partial^2 f}{\partial y \partial x}(0,0.2) \end{cases}, \quad \begin{cases} P(0.1,0) = f(0.1,0) \\ \frac{\partial^2 P}{\partial y^2}(0.1,0) = \frac{\partial^2 f}{\partial y^2}(0.1,0) \\ \frac{\partial P}{\partial y}(0.1,0.2) = \frac{\partial f}{\partial y}(0.1,0.2) \\ \frac{\partial^2 P}{\partial y^2}(0.1,0.2) = \frac{\partial^2 f}{\partial y^2}(0.1,0.2) \\ \frac{\partial P}{\partial y}(0.1,0.5) = \frac{\partial f}{\partial y}(0.1,0.5) \end{cases}$$

$$\begin{cases} \frac{\partial P}{\partial x}(0.9,0) = \frac{\partial f}{\partial x}(0.9,0) \\ \frac{\partial^2 P}{\partial y \partial x}(0.9,0.3) = \frac{\partial^2 f}{\partial y \partial x}(0.9,0.3) \\ \frac{\partial^3 P}{\partial y^2 \partial x}(0.9,0.6) = \frac{\partial^3 f}{\partial y^2 \partial x}(0.9,0.6) \\ \frac{\partial^4 P}{\partial y^3 \partial x}(0.9,0.9) = \frac{\partial^4 f}{\partial y^3 \partial x}(0.9,0.9) \end{cases}, \quad \begin{cases} \frac{\partial^2 P}{\partial x^2}(0.9,0.8) = \frac{\partial^2 f}{\partial x^2}(0.9,0.8) \\ \frac{\partial^2 P}{\partial x^2}(0.9,0.1) = \frac{\partial^2 f}{\partial x^2}(0.9,0.1) \end{cases},$$

$$\begin{cases} P(1,0) = f(1,0) \\ \frac{\partial P}{\partial y}(1,1) = \frac{\partial f}{\partial y}(1,1) \end{cases}$$

By the conditions above, the algebraic system of coefficients of p(x,y) is as follows:

$$\begin{cases} 2a_{1,2} = 0 \\ a_{1,0} + 0.1a_{1,1} + 0.01a_{1,2} + 0.001a_{1,3} = 0.1 \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ 1 + a_{0,1} + a_{2,1} + 3a_{0,3} + a_{3,1} + 4a_{0,4} + a_{4,1} = e \end{cases}$$

Therefore, the solution of system is
$a_{1,0} = a_{0,2} = a_{1,2} = a_{0,3} = a_{2,2} = a_{1,3} = a_{0,4} = a_{2,3} = 0, a_{1,1} = 1,$
$a_{0,0} = 3.46 \times 10^{-7}, \quad a_{0,1} = 0.999928024, \quad a_{4,1} = 6.678685615,$
$a_{3,1} = 0.13497021, \quad a_{2,1} = 0.510326531, \quad a_{2,0} = -1.314 \times 10^{-5},$
$a_{4,0} = -1.5909 \times 10^{-5}, \quad a_{3,0} = 2.8702 \times 10^{-5}$
Thus, the Birkhoff polynomial P is
$P_B(x,y) = 0.0000003459603111 + 0.999928024y - 0.00001314x^2 +$
$+xy + 0.000028702x^3 + 0.510326531x^2y -$
$-0.000015909x^4 + 0.13497021x^3y + 6.678685615x^4y.$
In the following example, the knots and Birkhoff conditions are given then, we obtain Birkhoff polynomial.

Example 3. By the following knots in $[0,1]^2$ and Birkhoff conditions and in view of Theorem 7 and the indices i,j,k,l, we have

$$\begin{cases} \frac{\partial^3 P}{\partial y^2 \partial x}(x_1,y_1) = \frac{\partial^3 f}{\partial y^2 \partial x}(x_1,y_1) \\ \frac{\partial P}{\partial x}(x_1,y_2) = \frac{\partial f}{\partial x}(x_1,y_2) \\ \frac{\partial^3 P}{\partial y^2 \partial x}(x_1,y_2) = \frac{\partial^3 f}{\partial y^2 \partial x}(x_1,y_2) \\ \frac{\partial^2 P}{\partial y \partial x}(x_1,y_3) = \frac{\partial^2 f}{\partial y \partial x}(x_1,y_3) \end{cases}, \quad \begin{cases} P(x_2,y_1) = f(x_2,y_1) \\ \frac{\partial^2 P}{\partial y^2}(x_2,y_1) = \frac{\partial^2 f}{\partial y^2}(x_2,y_1) \\ \frac{\partial P}{\partial y}(x_2,y_2) = \frac{\partial f}{\partial y}(x_2,y_2) \\ \frac{\partial^2 P}{\partial y^2}(x_2,y_2) = \frac{\partial^2 f}{\partial y^2}(x_2,y_2) \\ \frac{\partial P}{\partial y}(x_2,y_3) = \frac{\partial f}{\partial y}(x_2,y_3) \end{cases}$$

$$\begin{cases} \frac{\partial P}{\partial x}(x_3, y_1) = \frac{\partial f}{\partial x}(x_3, y_1) \\ \frac{\partial^2 P}{\partial y \partial x}(x_3, y_2) = \frac{\partial^2 f}{\partial y \partial x}(x_3, y_2) \\ \frac{\partial^3 P}{\partial y^2 \partial x}(x_3, y_3) = \frac{\partial^3 f}{\partial y^2 \partial x}(x_3, y_3) \\ \frac{\partial^4 P}{\partial y^3 \partial x}(x_3, y_4) = \frac{\partial^4 f}{\partial y^3 \partial x}(x_3, y_4) \end{cases}, \quad \begin{cases} \frac{\partial^2 P}{\partial x^2}(x_3, y_1) = \frac{\partial^2 f}{\partial x^2}(x_3, y_1) \\ \frac{\partial^2 P}{\partial x^2}(x_3, y_2) = \frac{\partial^2 f}{\partial x^2}(x_3, y_2) \end{cases},$$

$$\begin{cases} P(x_4, y_1) = f(x_4, y_1) \\ \frac{\partial P}{\partial y}(x_4, y_2) = \frac{\partial f}{\partial y}(x_4, y_2) \end{cases}$$

Now, we consider the points (11) in $[0,1]^2$, then

$$Z = \{(1,1), (2,0), (3,1), (3,2), (4,0)\}.$$

Regularity of $E_{i,k}$'s is obvious here:

$$E_{1,1} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}, \quad E_{2,0} = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{Vmatrix}, \quad E_{3,1} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

$$E_{3,2} = \begin{Vmatrix} 1 & 0 \\ 1 & 0 \end{Vmatrix}, \quad E_{4,0} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

Therefore,

$$a_{1,1} = 3, \quad N_{1,1} = 3, \quad a_{2,0} = 3, \quad N_{2,0} = 4$$

$$a_{3,1} = 4, \quad N_{3,1} = 3, \quad a_{3,2} = 2, \quad N_{3,2} = 1$$

$$a_{4,0} = 2, \quad N_{4,0} = 1$$

and also

$$N_1 = N_{2,0} = 4, \quad M_1 = 0,$$

$$N_2 = N_{1,1} = N_{3,1} = 3, \quad M_2 = 2$$

$$N_3 = N_{3,2} = N_{4,0} = 1, \quad M_3 = 4.$$

Using (9), we can write incidence matrix $\varepsilon_{4,4}$ in (10).
Thus, matrices

$$E_1 = \begin{Vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Vmatrix}, \quad E_2 = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix}, \quad E_3 = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

for knots x_1, x_2, x_3, x_4 and also the incidence matrices $E_{1,1}, E_{2,0}, E_{3,1}, E_{3,2}, E_{4,0}$ for knots $y_{i,k,j}$'s are regular. Thus by corollary 3.4 of [7, P.27], $\varepsilon_{4,4}$ is regular. For every $f \in C^5([0,1]^2)$ there exists $P \in \sum_{s=1}^{3} \Pi_{M_s} \otimes \Pi_{N_s}$ so that it satisfies interpolation conditions. Finally, with knots $(x_i, y_{i,k,j})$ in (11), we can establish P.

3.3. **Bivariate Haar Approximation.** In [9] and [15], the authors presented univariate Haar series. Now, we investigate a new case of bivariate Haar approximation in the following example.

Example 4. Using the approximation presented in [4], we compute Haar – Fourier coefficients

$$a_{m,n}(f) = \int_0^1 \int_0^1 f(x,y)\chi_{m,n}(x,y)dxdy$$

for bivariate function $f(x,y) = ye^x$ then, by (20) in [4], we have

$$a_{1,1}(f) = \frac{e-1}{2}, \quad a_{2,2}(f) = \frac{2e^{1/2} - e - 1}{2},$$

$$a_{3,3}(f) = \frac{2e^{1/4} - e^{1/2} - 1}{4}, \quad a_{3,4}(f) = \frac{6e^{1/4} - 3e^{1/2} - 3}{4},$$

$$a_{4,3}(f) = \frac{2e^{3/4} - e^{1/2} - e}{4}, \quad a_{4,4}(f) = \frac{6e^{3/4} - 3e^{1/2} - 3e}{4}.$$

We recall that the Haar function is given

$$\chi_{m,n}(x,y) = \begin{cases} 2^k & x \in \Delta_m^+, y \in \Delta_n \\ -2^k & x \in \Delta_m^-, y \in \Delta_n \\ 0 & (x,y) \notin \overline{\Delta_{n,m}} \end{cases}$$

where $\chi_{1,1} \equiv 1$ and the binary interval Δ_n and other signs in Definition 10 are satisfied. Then the Haar polynomial is:

$$P_H(x,y) = 2.218281828 - 0.210419644\chi_{2,2}(x,y) + 1.919329563\chi_{3,3}(x,y)+$$

$$+0.486540953\chi_{3,4}(x,y) - 0.033250766\chi_{4,3}(x,y) - 0.099752299\chi_{4,4}(x,y).$$

4. Comparison of function $f(x,y) = ye^x$ with $P_B(x,y)$ and $P_H(x,y)$

Using the following eight points, we compare $f(x,y), P_B(x,y), P_H(x,y)$

Tabl. 2. comparison of $f(x,y), P_B(x,y), P_H(x,y)$

(x,y)	$f(x,y)$	$p_B(x,y)$	$p_H(x,y)$
(0,0)	0	0.0000003	5.846521
(0,0.1)	0.1	0.099993	5.846521
(0.1,0)	0	0.0000002	5.846521
(0.1,0.1)	0.110517	0.110583	5.846521
(0.2,0.5)	0.610701	0.616053	4.413732
(0.9,0.1)	0.245960	0.679357	2.495203
(0.2,0.9)	1.099262	1.108896	2.980944
(0.9,0.9)	2.213642	6.114214	2.628206

Now, we compare $f(x, y), P_B(x, y)$ by using their graphs.

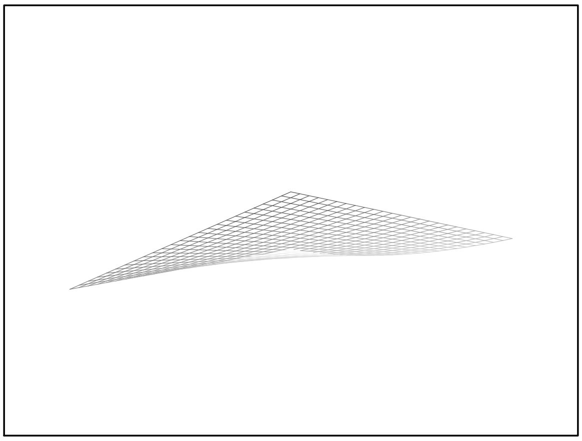

FIG. 5. The graph of $f(x, y)$ on $[0, 1]^2$

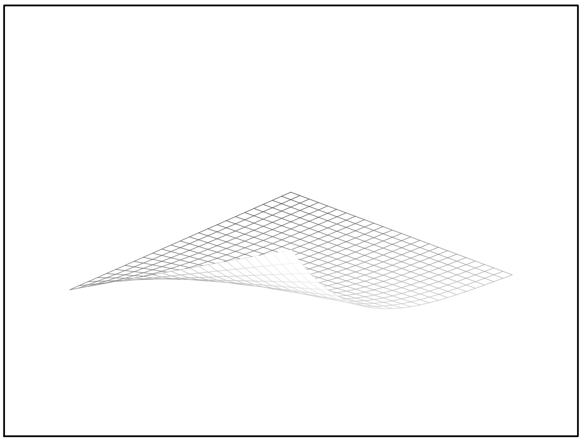

FIG. 6. The graph of $P_B(x, y)$ on $[0, 1]^2$

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