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On approach to determine the internal potential and gravitational energy of ellipsoid

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Formulas are derived for the calculation of the potential of bodies, which surface is a sphere or an ellipsoid, and the distribution function has a special form: a piecewise continuous one-dimensional function and a three-dimensional mass distribution. For each of these cases, formulas to calculate both external and internal potentials are derived. With their help, further the expressions are given for calculation of the potential (gravitational) energy of the masses of such bodies and their corresponding distributions. For spherical bodies, the exact and approximate relations for determining the energy are provided, which makes it possible to compare the iterative process and the possibility of its application to an ellipsoid. The described technique has been tested by a specific numerical example.

Keywords: *internal potential, gravitational energy, Cauchy formula, expansion coefficient.*

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1. Introduction

Determination of the potential outside and inside an ellipsoid is an important problem for a range of problems in applied and fundamental sciences. For example, the energy of plasma clot of ellipsoidal form [1] is determined by the corresponding formula for the energy of the ellipsoid, which takes into account the expression for the potential. It is important to study this issue and its further development for astronomy, because the figure of a celestial body is interpreted as the rotation of an ideal liquid body and is often taken as an ellipsoid of rotation [2]. A significant role is played by clarifying the conditions under which the planet is in a state of hydrostatic equilibrium or deviations from this state, which makes it possible to study dynamic changes within the planet [3]. It should be noted that a number of eminent scientists were engaged in these issues, ranging from Newton [4] to modern ones, for example, Chandrasekhar [2]. The object of study was both the figure and the inner content inside the planet [5]. For example, it was established in [6] that the minimum of E is attained for a homogeneous ball. Therefore, any progress in this research gives new opportunities in solving applied problems of physics, astronomy, geophysics and geodesy [7].

2. Statement of the problem

To determine the potentials (internal and external) with an inhomogeneous mass distribution, which is represented by the expansion in series, and using it, to calculate the gravitational energy. In this case, the external gravitational field should correspond to the real one, which is described by expansion in spherical functions and is given by a set of the series coefficients [8].

3. Main results

Let $\tau\colon \left\{\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\leqslant 1\right\}$ be an ellipsoidal body with a piecewise continuous three-dimensional function $\delta\in L_\tau^2$, which can be represented as a series [1]:

$$\delta(x_1,x_2,x_3)=\sum_{m+n+k=0}^{\infty} b_{mnk}W_{mnk}(x_1,x_2,x_3), \tag{1}$$

where $\{W_{mnk}\}$, $\{\omega_{mnk}\}$ are two biorthogonal polynomial systems in an ellipsoid,

$$b_{mnk}=\frac{\displaystyle\int_\tau \delta\omega_{mnk}\,d\tau}{\displaystyle\int_\tau W_{mnk}\,\omega_{mnk}\,d\tau}$$

are expansion coefficients in the system $\{W_{mnk}\}$.

To accelerate the convergence of the series (1), we represent it by the sum:

$$\delta(x_1,x_2,x_3)=\delta^0(\rho)+\sum_{m+n+k=0}^{\infty} b_{mnk}W_{mnk}(x_1,x_2,x_3), \tag{2}$$

assuming that the discontinuities of the function are concentrated on "concentric" ellipsoids with relative radii ρ_j, i.e.

$$\delta^0(\rho)=\begin{cases}\sum\limits_{j=0}^{mk} a_{1,j}\,\rho^j, & 0\leqslant\rho\leqslant\rho_1;\\ \sum\limits_{j=0}^{mk} a_{2,j}\,\rho^j, & \rho_1<\rho\leqslant\rho_2;\\ \dots & \\ \sum\limits_{j=0}^{mk} a_{n,j}\,\rho^j, & \rho_{nk-1}<\rho\leqslant\rho_{nk}.\end{cases} \tag{3}$$

In this representation, the series (2) is convergent on average [9]:

$$\lim_{N\to\infty}\int_\tau\left[\delta-\delta^0(\rho)-\sum_{m+n+k=0}^{N} b_{mnk}W_{mnk}(x_1,x_2,x_3)\right]^2 d\tau=0, \tag{4}$$

which guarantees uniform convergence of the following series [10]:

$$U(P)=U_0(P)+\sum_{m+n+k=0}^{N} b_{mnk}U_{mnk}(P),\quad P(x_1,x_2,x_3)\in\mathbb{R}^3,\quad Q(\xi,\eta,\zeta)\in\tau, \tag{5}$$

where

$$\text{a) } U(P)=G\int_\tau\frac{\delta(Q)}{r(Q,P)}\,d\tau_Q,\ \text{ b) } U_0(P)=G\int_\tau\frac{\delta^0(\rho)}{r(Q,P)}\,d\tau_Q,\ \text{ c) } U_{mnk}(P)=G\int_\tau\frac{W_{mnk}(Q)}{r(Q,P)}\,d\tau_Q, \tag{6}$$

From Bunyakovsky–Cauchy inequality follows:

$$\left|U(P)-U_0(P)-\sum_{m+n+k=0}^{N} b_{mnk}U_{mnk}(P)\right|^2=\left|\int_\tau\frac{\delta-\delta^0(\rho)-\sum_{m+n+k=0}^{N} b_{mnk}W_{mnk}(x_1,x_2,x_3)}{r(Q,P)}d\tau_Q\right|^2$$

$$\leqslant \int_\tau \left[\delta - \delta^0(\rho) - \sum_{m+n+k=0}^{N} b_{mnk} W_{mnk}(x_1, x_2, x_3)\right]^2 d\tau \int_\tau \frac{1}{r^2(Q,P)} d\tau_Q. \quad (7)$$

Since the second integral can be estimated as $\int_\tau \frac{1}{r^2(Q,P)} d\tau_Q \leqslant M$ [11], then (7) will take the form:

$$\left| U(P) - U_0(P) - \sum_{m+n+k=0}^{N} b_{mnk} U_{mnk}(P) \right|^2$$

$$\leqslant M \int_\tau \left[\delta - \delta^0(\rho) - \sum_{m+n+k=0}^{N} b_{mnk} W_{mnk}(x_1, x_2, x_3)\right]^2 d\tau \to 0, \quad (8)$$

whence the series converges uniformly. This gives the opportunity to calculate the internal potential, and further the potential (gravitational) energy using the known expansion coefficients (1). Moreover, each set $\{b_{mnk}\}$ defines its own function, and covers the entire class of piecewise continuous functions. Therefore, the series (1) converges on average [9]. To determine the potential, it becomes necessary to evaluate expressions (6). The function under the integral is a generalized Legendre polynomial of three variables and is defined as follows [1]:

$$W_{mnk} = \frac{1}{m!\,n!\,k!\,2^N} \frac{\partial^N}{\partial x_1^m \partial x_2^n \partial x_3^k} \left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 1 \right)^N. \quad (9)$$

Classical potential theory allows us to establish a formula for finding the potential of such a distribution [12].

The potential of the ellipsoid τ for the function $\left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} - 1\right)^N = (-1)^N \left(1 - \rho^2\right)^N$ is defined as [12]:

$$U = G \int \frac{(1-\rho^2)^N (-1)^N}{r(Q,P)} d\tau \frac{3 V_e G (-1)^N}{4 m!\,n!\,k!\,2^N (N+1)} \int_0^\infty \left(1 - \frac{x_1^2}{a_1^2+u} - \frac{x_2^2}{a_2^2+u} - \frac{x_3^2}{a_3^2+u}\right)^{N+1} \frac{du}{Q(u)}.$$

To determine the expression c) from the relations (6), we calculate

$$U_{mnk}(P) = G \int_\tau \frac{W(\rho)}{r(Q,P)} d\tau_Q = \frac{G}{m!\,n!\,k!\,2^N} \int_\tau \frac{1}{r(Q,P)} \frac{\partial^N}{\partial \xi^m \partial \eta^n \partial \zeta^k} \left(\frac{\xi^2}{a_1^2} + \frac{\eta^2}{a_2^2} + \frac{\zeta^2}{a_3^2} - 1 \right)^N d\tau \quad (10)$$

and use the property [12]:

$$\int_\tau \frac{1}{r(Q,P)} \frac{\partial^N f}{\partial \xi^m \partial \eta^n \partial \zeta^k} d\tau_Q = \frac{\partial^N}{\partial x_1^m \partial x_2^n \partial x_3^k} \left(\int_\tau \frac{f}{r(Q,P)} d\tau \right), \quad (11)$$

provided that $f\big|_\Omega = 0$, Ω is a body surface.

Taking into account (11) and ($f = (1-\rho^2)^N$), the potential c) from (6) can be represented as [6]:

$$U_{mnk} = \frac{3 V_e (-1)^N}{4 m!\,n!\,k!\,2^N (N+1)} \frac{\partial^N}{\partial x_1^m \partial x_2^n \partial x_3^k} \int_0^\infty \left(1 - \frac{x_1^2}{a_1^2+u} - \frac{x_2^2}{a_2^2+u} - \frac{x_3^2}{a_3^2+u}\right)^{N+1} \frac{du}{Q(u)}. \quad (12)$$

Potential (12) can be written as

$$U_{mnk} = \frac{3 V_e (-1)^N N!}{4 m!\,n!\,k!\,2^N} \sum_{l=0}^{N+1} \frac{(-1)^l}{(N+1-l)!\,l!} \frac{\partial^N}{\partial x_1^m \partial x_2^n \partial x_3^k} \int_0^\infty (\mu^2)^l du$$

$$= \frac{3V_e(-1)^N N!}{4m!\, n!\, k!\, 2^N} \sum_{l=0}^{N+1} \frac{(-1)^l 2^l}{(N+1-l)!} \sum_{t_1+t_2+t_3=l} \frac{(2t_1-1)!!\,(2t_2-1)!!\,(2t_3-1)!!\, M_{t_1t_2t_3}}{(2t_1-m)!\,(2t_2-n)!\,(2t_1-k)!}$$
$$\times \left(\left(\frac{x_1}{a_1}\right)^{2t_1-m} \left(\frac{x_2}{a_2}\right)^{2t_2-n} \left(\frac{x_3}{a_3}\right)^{2t_3-k} \right),$$

where $\mu^2 = \frac{x_1^2}{a_1^2+u} + \frac{x_2^2}{a_2^2+u} + \frac{x_3^2}{a_3^2+u}$, $Q(u) = \sqrt{(a_1^2+u)(a_2^2+u)(a_3^2+u)}$.

We also have

$$M_{t_1t_2t_3} = a_1^{2t_1} a_2^{2t_2} a_3^{2t_3} \left(\int_0^\infty \frac{du}{(a_1^2+u)^{t_1}(a_2^2+u)^{t_2}(a_3^2+u)^{t_3} Q(u)} \right). \tag{13}$$

Calculations of values (13) using recurrences [13, 14] are unstable. Therefore, we use the expansion of the function under the integral in binomial series with further multiplication and integration of these series. To do this, in (13) we make a change of variables:

$$z^2 = a_1^2 + u, \quad 2zdz = du, \quad u_0 = 0, \quad z_0 = a_1, \quad u_1 = \infty, \quad z_1 = \infty.$$

Then

$$M_{t_1t_2t_3} = a_1^{2t_1} a_2^{2t_2} a_3^{2t_3} \left(\int_{a_1}^\infty \frac{2dz}{z^{2t_1+1}(a_2^2 - a_1^2 + z^2)^{t_2+\frac{1}{2}}(a_2^2 - a_1^2 + z^2)^{t_3+\frac{1}{2}}} \right),$$

also

$$\gamma^{2t_3}\beta^{2t_2} \left(\int_{a_1}^\infty \frac{2dz}{z^{2t_1+1}(e_1^2+z^2)^{t_2+\frac{1}{2}}(e_2^2+z^2)^{t_3+\frac{1}{2}}} \right), \quad \gamma = \frac{a_1}{a_3}, \quad \beta = \frac{a_2}{a_3}, \quad e_1^2 = \beta^2 - 1, \quad e_2^2 = \gamma^2 - 1.$$

Let expand the integrands in series

$$\frac{1}{(e_1^2+z^2)^{t_2+\frac{1}{2}}} = \frac{1}{z^{2t_2+1}\left(1+\frac{e_1^2}{z^2}\right)^{t_2+\frac{1}{2}}} = \sum_{l=0}^\infty \frac{(-1)^l (2t_2+2l-1)!!\, e_1^{2l}}{2^l (2t_2-1)!!\, z^{2t_2+2l+1}} = \sum_{l=0}^\infty \frac{b_l}{z^{2t_2+2l+1}},$$
$$\frac{1}{(e_2^2+z^2)^{t_3+\frac{1}{2}}} = \frac{1}{z^{2t_3+1}\left(1+\frac{e_2^2}{z^2}\right)^{t_2+\frac{1}{2}}} = \sum_{l=0}^\infty \frac{(-1)^l (2t_3+2l-1)!!\, e_2^{2l}}{2^l (2t_3-1)!!\, z^{2t_3+2l+1}} = \sum_{l=0}^\infty \frac{c_l}{z^{2t_3+2l+1}}.$$

The series converge provided that $a_1 > a_2 > a_3$, so

$$\left|\frac{a_3^2-a_1^2}{z^2}\right| < \left|\frac{a_3^2-a_1^2}{a_1^2}\right| = \left|1-\gamma^2\right| < 1, \quad \left|\frac{a_2^2-a_1^2}{z^2}\right| < \left|\frac{a_2^2-a_1^2}{a_1^2}\right| = \left|1-\beta^2\right| < 1.$$

The expression under the integral is the product of two convergent series, whose coefficients at degrees are determined by the scheme:

$$d_0 = b_0c_0, \quad d_1 = b_1c_0 + b_0c_1, \quad d_i = \sum_{l=0}^{i} c_l b_{i-l}, \tag{14}$$

that is, the corresponding power series is

$$\sum_{l=0}^\infty \frac{d_l}{z^{2t_1+2t_2+2t_3+2l+1}}. \tag{15}$$

Substituting (14) in (13) and evaluating the integral, we obtained

$$M_{t_1t_2t_3} = -\sum_{l=0}^{\infty} \frac{d_l}{(2t_1+2t_2+2t_3+2l)z^{2t_1+2t_2+2t_3+2l}}\bigg|_{a_1}^{\infty} = \sum_{l=0}^{\infty} \frac{d_l}{(2t_1+2t_2+2t_3+2l)a_1^{2t_1+2t_2+2t_3+2l}}.$$

This allows us to determine the value of c) in (6).

Expression b) in (6) is the one-dimensional distribution potential. For spherical bodies this potential can be represented analytically by the formula (22). Even for homogeneous bodies, there are no such simple formulas for an ellipsoid, which is explained primarily by the dependence of the potential on three coordinates. Therefore, further we use an approximation approach, we represent the function $\delta^0(\rho)$ as a series in Legendre polynomials of even orders, extending the function to the interval $-1 \leqslant \rho \leqslant 1$.

Now, the representation of the potential is [15]

$$\tilde{\delta}(\rho) = \sum_{n=0}^{\infty} c_{2n} P_{2n}(\rho), \tag{16}$$

where

$$P_n(\rho) = \frac{1}{2^n n!}\frac{d^n\left(\rho^2-1\right)^n}{d\rho^n} \tag{17}$$

is Legendre polynomial,

$$c_{2n} = (4n+1)\int_{-1}^{1} \delta(\rho)P_{2n}(\rho)d\rho = (4n+1)\int_{0}^{1} \delta(\rho)P_{2n}(\rho)d\rho \tag{18}$$

are expansion coefficients.

The expression of the potential presented in (6) by clause a) can be written as

$$U_0(P) = G\int_\tau \frac{\delta(\rho)}{r}\,d\tau = G\sum_{n=0}^{\infty} c_{2n}\int_\tau \frac{P_{2n}(\rho)}{r}d\tau = \sum_{n=0}^{\infty} c_{2n}U_{2n}, \tag{19}$$

where

$$U_{2n} = G\sum_{t=0}^{n} d_{2n}^{t}\left(\frac{u_2}{t+1} - u_{2t}\right),$$

$$u_{2t} = \int_\tau \frac{\rho^{2t}}{r}d\tau = \frac{3V_e t!}{4}\sum_{l=0}^{t}\frac{(-1)^{t-l}}{(t-l)!}\sum_{t_1+t_2+t_3=l}\frac{x_1^{2t_1}x_2^{2t_2}x_3^{2t_3}}{t_1!\,t_2!\,t_3!\,a_1^{2t_1}a_2^{2t_2}a_3^{2t_3}}M_{t_1t_2t_3},$$

and for this class of functions, convergence on average is guaranteed [14, 16].

Thus, we can find the potential inside an ellipsoidal body and proceed to study the gravitational energy, which is determined as

$$E = -\frac{1}{2}\int_\tau U\delta d\tau \tag{20}$$

where U is a potential inside a body τ generated by the distribution δ, where τ is the area of integration.

Substituting (1) and (5), equality (20) takes the form

$$E = -\frac{1}{2}\bigg(\int_\tau U_0\delta^0 d\tau + \sum_{m+n+k=0}^{N} b_{mnk}\Big(\int_\tau \left(U_{mnk}\delta^0 + W_{mnk}U_0\right)d\tau + \sum_{m_1+n_1+k_1=0}^{\infty}\sum_{m+n+k=0}^{\infty} b_{m_1n_1k_1}\int_\tau W_{m_1n_1k_1}U_{mnk}d\tau\Big)\bigg). \tag{21}$$

To calculate the components of the right side (21), first, we find

$$\int_\tau W_{m_1n_1k_1} U_{mnk} d\tau = \frac{1}{m!\,n!\,k!\,2^{N+N_1} m_1!\,n_1!\,k_1!\,(N+1)} \times \int_\tau \frac{\partial^{N+N_1}}{\partial x_1^{m+m_1} \partial x_2^{n+n_1} \partial x_3^{k+k_1}} \left(\int_0^\infty \left(\frac{x_1^2}{a_1^2+u} + \frac{x_2^2}{a_2^2+u} + \frac{x_3^2}{a_3^2+u} - 1 \right)^{N+1} du \right) (\rho^2-1)^{N_1} d\tau,$$

so

$$\int_\tau W_{m_1n_1k_1} U_{mnk} d\tau = \frac{3V_e(-1)^N N!!2^{l-N}}{4m!n!k!m_1!n_1!k_1!} \sum_{l=0}^{N+1} \frac{(-1)^{N-l+1}}{(N+1-l)!(N_1-N+2l+3)!!} \frac{\partial^N}{\partial x_1^{m'} \partial x_2^{n'} \partial x_3^{k'}} \int_0^\infty (\mu^2)^l \frac{du}{Q(u)} (\rho^2-1)^{N_1} d\tau$$

$$= \frac{3V_e(-1)^N N!N_1!2^{l-N}}{4m!n!k!m_1!n_1!k_1!} \sum_{l=0}^{N+1} \frac{(-1)^{N-l+1}}{(N+1-l)!(N_1-N+2l+3)!!} \times \sum_{t_1+t_2+t_3=l} \frac{(2t_1-1)!!(2t_2-1)!!(2t_3-1)!!M_{t_1t_2t_3}}{(2t_1-m)!!(2t_2-n)!!(2t_3-k)!!},$$

$$m' = m+m_1, \quad n' = n+n_1, \quad k' = k+k_1, \quad N' = N+N_1.$$

Similarly,

$$\int_\tau \delta^0(\rho) U_{mnk} d\tau = \frac{(-1)^N N!}{4m!\,n!\,k!\,2^N} \sum_{l=0}^{N+1} \frac{(-1)^l}{(N+1-l)!l!} \frac{\partial^N}{\partial x_1^m \partial x_2^n \partial x_3^k} \int_\tau \left(\int_0^\infty (\mu^2)^l du \right) \delta^0(\rho) d\tau$$

$$= \frac{(-1)^N N!}{4m!\,n!\,k!\,2^N} \sum_{l=0}^{N+1} \frac{(-1)^l}{(N+1-l)!} K'_{mnk},$$

where

$$K'_{mnk} = \frac{2^l N!}{(2l-N+1)!!} \sum_{t_1+t_2+t_3=l} \frac{(2t_1-1)!!\,(2t_2-1)!!\,(2t_3-1)!!\,M_{t_1t_2t_3}}{(2t_1-m)!!\,(2t_2-n)!!\,(2t_3-k)!!} R_{2l-N+2},$$

$$R_s = \int_0^1 \delta^0(\rho) \rho^s d\rho.$$

Therefore,

$$\int_\tau U^0 \delta^0(\rho) d\tau = \sum_{n=0}^\infty \sum_{m=0}^\infty r_{2n} r_{2m} \int_\tau U_{2n} P_{2m} d\tau.$$

In addition,

$$\int_\tau U_{2n} P_{2m} d\tau = \frac{3V_e}{2^{2m}(2m)!} \int_\tau \frac{d^{2m}}{d\rho^{2m}} \left(\rho^2-1\right)^{2m} \sum_{t=0}^n d_{2n}^t (u_2 - u_{2t+2})\, d\tau$$

$$= \frac{3V_e}{2^{2m}(2m)!} \sum_{t=0}^n d_{2n}^t \int_0^1 \frac{d^{2m}}{d\rho^{2m}} \left(\rho^2-1\right)^{2m} \left(\rho^2 \chi_2 - \rho^{2t+2} \chi_{2t+2}\right) d\rho$$

$$= \frac{3V_e}{4} \sum_{t=0}^n d_{2n}^t \left(\frac{\chi_2}{(2m+5)!!} - \frac{(2t+3)!!}{(2m+2t+5)!!} \chi_{2t+2} \right),$$

$$\chi_{2l} = \frac{\rho^{2l+2} l!}{(2l+1)!!} \sum_{t_1+t_2+t_3=t} \frac{(2t_1-1)!!\,(2t_2-1)!!\,(2t_3-1)!!}{t_1!\,t_2!\,t_3!} M_{t_1 t_2 t_3},$$

$$U_0 = \int_\tau \frac{\delta^0(\rho)}{r} d\tau = \sum_{n=0}^{\infty} c_n \int_\tau \frac{P_{2n}(\rho)}{r} d\tau = \sum_{n=0}^{\infty} c_n \sum_{l=0}^{n} d_n^l u_t,$$

where

$$u_t = \int_\tau \frac{\rho^2 t}{r} d\tau = \frac{3V_e t!}{4} \sum_{l=0}^{t} \frac{(-1)^{t-1}}{(t-l)!} \sum_{t_1+t_2+t_3=l} \frac{x_1^{2t_1} x_2^{2t_2} x_3^{2t_3}}{t_1!\,t_2!\,t_3!\,a_1^{2t_1} a_2^{2t_2} a_3^{2t_3}} M_{t_1 t_2 t_3}.$$

Using the previous relations, we obtain

$$\int_\tau W_{mnk} u_t \, d\tau = \frac{3V_e t!}{4m!\,n!\,k!} \sum_{l=0}^{t} \frac{(-1)^{t-l}}{(t-l)!} \sum_{t_1+t_2+t_3=l} \frac{2^l (2t_1-1)!!\,(2t_2-1)!!\,(2t_3-1)!!}{(2t_1-m)!!\,(2t_2-n)!!\,(2t_3-k)!!\,(2l+N+3)!!} M_{t_1 t_2 t_3},$$

and, finally, for even m, n, k, we have

$$\int_\tau W_{mnk}\, U_0 \, d\tau = \sum_{n=0}^{\infty} c_n \sum_{l=0}^{n} d_n^l \int_\tau W_{mnk} u_t \, d\tau.$$

4. The case of the globular planet

For a piecewise continuous function $\delta^0(\rho)$ represented by the expression (3) in the case of a sphere, the potential in the interlayer τ_i $\{\rho_{i-1} \leqslant \rho \leqslant \rho_i\}$ is defined

$$w_i = \frac{M_{i-1} - \sum_{j=0}^{m} \frac{a_{i,j} \rho_i^{j+2}}{j+2}}{\rho} + M_i' + \sum_{j=0}^{m} \frac{a_{i,j}}{j+2} \left(\rho_i^{j+2} - \frac{\rho^{j+2}}{j+3} \right), \tag{22}$$

where M_{i-1} is the mass of the body $\tau_1 \cup \tau_2 \cup \ldots \cup \tau_{i-1}$, M_{i-1}' is the constant potential inside the body $\tau_1 \cup \tau_2 \cup \ldots \cup \tau_{i-1}$, which are determined by the sum of the interlayers $\tau_{i+1} \cup \ldots \cup \tau_m$.

So, using this potential, we can determine the energy of the i-th layer

$$E_i = \left(M_{i-1} - \sum_{j=0}^{m} \frac{a_{i,j} \rho_i^{j+2}}{j+2} \right) \sum_{l=0}^{mk} \frac{a_{i,t}(\rho_i^{2l+2} - \rho_{i-1}^{2l+2})}{2l+2}$$
$$+ M_i' \sum_{l=0}^{n} \frac{a_{i,t}(\rho_i^{2l+3} - \rho_{i-1}^{2l+3})}{2l+3} - \sum_{j=0}^{mk} \sum_{l=0}^{mk} \frac{a_{i,j} d_{2n}^l (\rho_i^{j+2l+5} - \rho_{i-1}^{j+2l+5})}{(j+2)(j+3)(j+2l+5)}. \tag{23}$$

Thus,

$$E = \sum_{i=1}^{nk} \int_{\tau_i} w_i \sum_{l=0}^{mk} a_{i,l} \rho^l d\tau = \sum_{i=1}^{nk} E_i. \tag{24}$$

The above formulas (22)–(24), which coincide in content with the results of the work [17], allow us to write down the relation for determining the gravitational energy in each layer as well for a one-dimensional model (model PREM [18]) as for its modification — an ellipsoid.

5. Some preliminary calculations, comparisons and analysis of the obtained results

Let us show the practical application of the above algorithms and formulas on a specific example, focusing on the calculation of potential energy. For this, we present the found energy values for the

Table 1. Exact and approximate energy values for a sphere and an ellipsoid.

Figure	Approximate energy values	Exact energy values	Amendment for inhomogeneity
Sphere	$-2.4692036680 \cdot 10^{39}$ ergs	$-2.4696997141 \cdot 10^{39}$ ergs	—
Ellipsoid	$-2.4270418572 \cdot 10^{39}$ ergs	—	$-1.989389474 \cdot 10^{39}$ ergs

Earth mass distribution function [18] accepted in geophysics for a spherical and ellipsoidal planet [19]. Also, complete the [19] results considering the three-dimensionality of the planet. Calculations are made for a three-dimensional mass distribution, taken from the work [8].

6. Conclusions

The given formulas of potential and energy of three-dimensional bodies of spherical and ellipsoidal shape differ from each other. For a sphere it is possible to compare calculations in two ways. The value of energy is determined approximately with the required accuracy, as evidenced by the results of Table 1 ($-2.4692036680 \cdot 10^{39}$ ergs and $-2.4696997141 \cdot 10^{39}$ ergs). For an ellipsoid, there are no exact formulas for calculating the energy, and therefore we used the values, which was found approximately. Analysis and comparison of this value ($-1.989389474 \cdot 10^{39}$ ergs) together with the correction for ellipsoidality ($-1.8108 \cdot 10^{39}$ ergs) showed that their influence on the final energy value is the same. Although the total energy value is three orders of magnitude higher, regional studies (for example, the movement of tectonic plates inside the Earth) should take into account both the ellipsoidal nature of the planet and its inhomogeneity.

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Підхід до визначення внутрішнього потенціалу та гравітаційної енергії еліпсоїда

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Встановлено формули для обчислення потенціалу тіл, поверхня яких є куля або еліпсоїд, а функція розподілу має спеціальний вигляд: кусково-неперервна одновимірна функція або тривимірний розподіл мас. Для кожного з цих випадків виведені формули для обчислення як зовнішнього, так і внутрішнього потенціалів. З їх допомогою далі подаються вирази для обчислення потенціальної (гравітаційної) енергії мас таких тіл та їх відповідних розподілів. Для тіл кульової форми подаються точні та наближені співвідношення визначення енергії, що дає можливість порівняння ітераційного процесу та можливість його застосування до еліпсоїда. Описана методика апробована на конкретному числовому прикладі.

Ключові слова: *внутрішній потенціал, гравітаційна енергія, формула Коші, коефіцієнти розкладу.*