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Enlarging the radius of convergence for Newton–like method in which the derivative is re-evaluated after certain steps

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Numerous attempts have been made to enlarge the radius of convergence for Newton–like method under the same set of conditions. It turns out that not only the radius of convergence but the error bounds on the distances involved and the uniqueness of the solution ball can more accurately be defined.

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1. Introduction

Let C be an open convex subset of Hilbert space B. The problem of computing solution s^* as of equation
$$F(x) = 0, \tag{1}$$
where $F\colon C \to B$ is differentiable in the sense of Fréchet is of extreme importance, since many applications reduce to solving (1). But closed form expression for s^* can be obtained only in special cases. That is why the solutions methods for equations (1) are iterative (mostly).

Recently a ball convergence result was given by Măruşter in [1,2] for Newton–like method defined by iteration function H
$$\begin{aligned} y_{m+1} &= y_m - F'(x_n)^{-1}F(y_m), \quad m = 1, 2, \dots, k-1, \quad y_1 = x_n, \\ x_{n+1} &= H(x_n) = x_n - F'(x_n)^{-1}\sum_{m=1}^{k} F(y_m), \quad n = 0, 1, \dots. \end{aligned} \tag{2}$$
This iteration can be considered as Picard—like one. Several well studied methods are special cases of (2).

Potra and Pták [3] (PP) studied (2) when $k = 2$. Moreover, in the scalar case Potra and Pták method was studied by Traub [4]. Ortega and Rheinboldt showed on infinite Euclidean space [5] that PP is of convergence order three. Notice also that PP is a special case of a multi-point method of the same order given in [6,7] by Ezquerro and Hernández (EH). Moreover, Hernández and Romero [8] provided the radius of convergence for (EH) method. Furthermore, Cătinaş [9] gave a radius of convergence for the general Picard iteration. Two-step methods for solving nonlinear problems were studied in [10,11, 14].

Finally, Măruşter [2] gave a ball convergence result for method (2). Motivated by Măruşter's paper we provided a ball convergence under the same conditions with benefits:

1) at least as large radius of convergence (so at least as many initial points become available),
2) tighter error or bounds on $\|x_n - s^*\|$ become available (so at least a few iterations are needed to obtain certain error tolerance),
3) at least as precise information on the where about of the solution s^* is given.

This technique was applied to other iterative methods [12,13,15].

The ball convergence is given in Section 2, whereas the numerical experiments in Section 3.

2. Ball convergence

The aforementioned benefits are based on certain types of Lipschitz conditions. From now on we assume that s^* is a simple solution of equation (1) and $F\colon C \to B$ is Fréchet differentiable.

Definition 1. *We say that operator F' satisfies the center–Lipschitz condition if there exists $l_0 > 0$ such that*

$$\|F'(y) - F'(s^*)\| \leqslant l_0 \|y - s^*\| \tag{3}$$

for all $y \in C$.

Set $C_0 = C \cap U(s^*, \frac{1}{b_0 l_0})$, where b_0 will be determined later.

Definition 2. *We say that operator F' satisfies the restricted–Lipschitz condition if there exists $l > 0$ such that*

$$\|F'(y) - F'(x)\| \leqslant l \|y - x\| \tag{4}$$

for all $x, y \in C_0$.

Definition 3. *We say that operator F' satisfies the Lipschitz condition if there exists $l_1 > 0$ such that*

$$\|F'(y) - F'(x)\| \leqslant l_1 \|y - x\| \tag{5}$$

for all $x, y \in C$.

Remark 1. It follows from these definitions

$$C_0 \subset C \tag{6}$$

that

$$l_0 \leqslant l_1 \tag{7}$$

and

$$l \leqslant l_1. \tag{8}$$

We shall assume from now on that

$$l_0 \leqslant l. \tag{9}$$

Otherwise l_0 can replace l in all results that follow.

An upper bound on $\|F'(x)^{-1}\|$ was determined in [2] using (5). Indeed, assume

$$\|F'(s^*)^{-1}\| \leqslant b_0. \tag{10}$$

Then, for $v \in U(s^*, \frac{1}{b_0 l_1})$ we get by (5) and (10) that

$$\|F'(s^*)^{-1}(F'(v) - F'(s^*))\| \leqslant b_0 l_1 \|v - s^*\| < b_0 l_1 \frac{1}{b_0 l_1} = 1$$

so $F'(v)^{-1} \in L(B, B)$ by a lemma attributed to Banach [5,16] linear operators and

$$\|F'(v)^{-1}\| \leqslant \frac{b_0}{1 - b_0 l_1 \|v - s^*\|}. \tag{11}$$

However, notice that the weaker (3) can be used to obtain instead of (5) the tighter estimate

$$\|F'(v)^{-1}\| \leqslant \frac{b_0}{1 - b_0 l_0 \|v - s^*\|}. \tag{12}$$

Let $\{r_n\}$, $n = 1, 2, \ldots, k$ be a scalar sequence defined by

$$\begin{aligned} r_1 &= \bar{r}, \\ r_{n+1} &= \bar{a} r_n \Big(1 + \frac{r_n}{2\bar{r}}\Big). \end{aligned} \tag{13}$$

Then if $\bar{a} > \frac{2}{3}$ and $\bar{r} > 0$, sequence $\{r_n\}$ is strictly increasing [2]. The ball convergence is based on conditions (A).

Suppose:

(A_1) sequence $\{r_n\}$ is generated by (13) for $\bar{a} = a = \sqrt{3} - 1$, $\bar{r} = r \leqslant \frac{a}{b_0(l+al_0)}$ and $U(s^*, r_k) \subset C$;

(A_2) conditions (3), (4) and (10) hold.

Theorem 1. *Suppose conditions (A) hold. Then, sequence $\{x_n\}$ generated by method (2) is well defined in $U(s^*, r_0)$, remains in $U(s^*, r_0)$ and converges to the unique solution $s^* \in U(s^*, r_0)$ of equation (1), where*

$$r_0 = \frac{a}{b_0(l + al_0)}. \tag{14}$$

Moreover, the rate of convergence is at least $k+1$, and for all $x \in U(s^*, r_0)$

$$\|H(x) - s^*\| \leqslant \left(\frac{b_0(l + al_0)}{a}\right)^k \|x - s^*\|^{k+1}. \tag{15}$$

Proof. Simply use the proof of Corollary 3.2 in [2, page 19] with l, r_0 replacing l_1,

$$\bar{r}_0 = \frac{a}{(1+a)b_0 l_1} \tag{16}$$

in [2], respectively.

We also use (12) instead of (11) and notice that

$$\frac{b_0}{l - b_0 l_0 r_0} = \frac{b_0(l + al_0)}{l}$$

by the definition of r_0. ■

Remark 2. It follows from (7), (8), (14) and (16) that

$$\bar{r}_0 \leqslant r_0. \tag{17}$$

The corresponding to (15) given in [2] is

$$\|H(x) - s^*\| \leqslant \left(\frac{b_0(1 + a)l_1}{a}\right)^k \|x - s^*\|^{k+1} \tag{18}$$

for all $x \in U(s^*, \bar{r}_0)$.

By completing (15) to $U(s^*, r_0)$ we see that new ratio of convergence is at least as small as the old one since

$$\frac{b_0(l + al_0)}{a} \leqslant \frac{b_0(1 + a)l_1}{a}.$$

Clearly, Theorem 1 reduces to Corollary 3.2 in [2] if $l_0 = l = l_1$.

In view of (17) the uniqueness ball has been extended from $U(s^*, \bar{r}_0)$ to $U(s^*, r_0)$.

It turns out that we can do even better.

Proposition 3. Suppose that there exists $R \geqslant r_0$ such that

$$l_0 R < 2. \tag{19}$$

Set $C_1 = C \cap U(s^*, R)$.

Then, the only solution of equation (1) in C_1 is s^*.

Proof. Let $z \in C_1$ with $F(z) = 0$. Define $T = \int_0^1 F'(s^* + \theta(z - s^*))d\theta$. Then, using (3) and (19), we get in turn that

$$\|F'(s^*)^{-1}(T - F'(s^*))\| \leqslant l_0 \int_0^1 \theta \|z - s^*\| d\theta \leqslant \frac{l_0 R}{2} < 1,$$

so $z = s^*$ by $T^{-1} \in L(B, B)$ and the identity $0 = F(s^*) - F(z) = T(s^* - z)$. ■

Remark 3. In view of the above the benefits as stated in the introduction have been justified. Clearly, our technique extends the results of the aforementioned methods along the same lines. The efficiency of method (2) given in [2] is also improved, since the number of steps k after which F' is re-evaluated periodically increases under our approach. Notice also that

$$\frac{r_0}{\bar{r}_0} = \frac{1+a}{1+a\frac{l_0}{l}} \to 1+a = \sqrt{3}$$

as $\frac{l_0}{l} \to 0$. Hence, own technique increases the ball of convergence by almost $\sqrt{3}$ times.

3. Numerical examples

In this section we give some examples to confirm the theoretical results, namely that (17) is satisfied.

Example 1. Let $C = U(1, 1-p)$, $p \in (0, 0.9)$ and $B = \mathbb{R}$. Define function F on C by

$$F(x) = x^3 - p. \tag{20}$$

Since $s^* = \sqrt[3]{p}$ and $F'(x) = 3x^2$, we get that $b_0 = \frac{1}{3\sqrt[3]{p^2}}$, $l_1 = 6(2-p)$, $l_0 = 3(2-p+\sqrt[3]{p})$ and $l = 6\min\left(2-p, s^* + \frac{1}{b_0 l_0}\right)$. Let $p = 0.725$. Then, we get $s^* \approx 0.8984$,

$$C = (0.7250, 1.2750), \quad b_0 \approx 0.4130, \quad l_1 = 7.6500, \quad l_0 \approx 6.5201,$$

$$C_0 \approx (0.7250, 1.2697), \quad l \approx 7.6181, \quad r_0 \approx 0.1430, \quad \bar{r}_0 \approx 0.1338.$$

So, all conditions in Remark 1 are satisfied.

Example 2. Let $C = U(0,1)$ and $B = \mathbb{R}^3$. Define function F on C for $x = (\xi_1, \xi_2, \xi_3)^T$ by

$$F(x) = \left(e^{\xi_1} - 1, \frac{e-1}{2}\xi_2^2 + \xi_2, \xi_3\right). \tag{21}$$

Since $s^* = (0,0,0)^T$ and $F'(x) = \operatorname{diag}\{e^{\xi_1}, (e-1)\xi_2 + 1, 1\}$, we get that $b_0 = 1$, $l_1 = e$, $l_0 = e-1$ and $l = \max\left(e^{\frac{1}{e-1}}, e-1\right)$, $r_0 \approx 0.2402, \bar{r}_0 \approx 0.1555$. So, all conditions in Remark 1 are satisfied for this example too.

Notice that our radius r_0 is larger than $\bar{r}_0$ used in [1] as expected, since $l_0 < l_1$ and $l < l_1$. Hence, the benefits as claimed in the Introduction are also numerically justified.

4. Conclusion

The convergence analysis of Newton–like method is provided under classical center and restricted Lipschitz conditions. As a result, the convergence ball and the ball of uniqueness of the solution are enlarged, and the limits of error at the distances involved are determine more accurately. Moreover, the results were obtained under the same set of conditions as in the previous work. Numerical results that confirm the theoretical ones are given.

[1] Măruşter Ş. Estimating local radius of convergence. Symposium "Symbolic and Numeric Algorithm for Scientific Computation" (SYNASC), Workshop Iteratime Approximation of Fixed Points, 24–27 Sept. 2016. West University of Timisoara, Timisoara, Romania (2016).

[2] Măruşter Ş. On the local convergence of the Modified Newton method. Annals of West University of Timisoara – Mathematics and Computer Science. **57** (1), 13–22 (2019).

[3] Potra F. A., Pták V. Nondiscrete induction and iterative processes. Pitman Publ., London (1984).

[4] Traub J. F. Iterative methods for the solution of equations. Chelsea Publishing Company, New York (1982).

[5] Ortega J. M., Rheinboldt W. C. Iterative solution of nonlinear equation in several variables. Acad. Press, New York (1970).

[6] Ezquerro J. A, Hernández M. A. An improvement of the region of accessibility of Chebyshev's method from Newton's method. Mathematics of Computation. **78** (267), 1613–1627 (2009).

[7] Ezquerro J. A, Hernández M. A. An optimization of Chebyshev's method. Journal of Complexity. **25** (4), 343–361 (2009).

[8] Hernández-Veron M. A., Romero N. On the local convergence of a third order family of iterative processes. Algorithms. **8**, 1121–1128 (2015).

[9] Cătinaş E. Estimating the radius of an attraction ball. Applied Mathematics Letters. **22**, 712–714 (2009).

[10] Iakymchuk R. P., Shakhno S. M., Yarmola H. P. Convergence analysis of a two-step modification of the Gauss–Newton method and its applications. Journal of Numerical and Applied Mathematics. **126**, 61–74 (2017).

[11] Magreñán Á. A., Argyros I. K. Two-step Newton methods. Journal of Complexity. **30** (4), 533–553 (2014).

[12] Argyros I. K., Shakhno S. Extending the Applicability of Two-Step Solvers for Solving Equations. Mathematics. **7** (1), 62 (2019).

[13] Argyros I. K., Magreñán Á. A. A Contemporary Study of Iterative Methods. Convergence, Dynamics and Applications. 333–346 (2018).

[14] Shakhno S. M. On an iterative algorithm with superquadratic convergence for solving nonlinear operator equations. Journal of Computational and Applied Mathematics. **231** (1), 222–235 (2009).

[15] Argyros I. K., Shakhno S., Yarmola H. Two-Step Solver for Nonlinear Equations. Symmetry. **11** (2), 128 (2019).

[16] Kantorovich L. V., Akilov G. P. Functional Analysis. Oxford, Pergamon (1982).

Збільшення радіусу збіжності методу типу Ньютона, в якому похідна обчислюється через декілька кроків

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Зроблено спробу збільшити радіус області збіжності методу типу Ньютона за тих же умов, за яких метод вивчався раніше. Аналіз збіжності проведено за центральних та обмежених умов Ліпшиця. Крім радіусу області збіжності, вдалося отримати точніші оцінки похибки, а також більший радіус області єдиності розв'язку. Ці переваги є чисельно обґрунтованими.

Ключові слова: *радіус збіжності, метод типу Ньютона, гільбертів простір.*