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Method of normal oscillations and substantiation of the choice of parameters for certain nonlinear systems with two degrees of freedom

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On the example of the plane model of wheeled vehicle oscillations with adaptive power characteristic of the suspension system, the methodology for selecting its main parameters that would maximize the movement smoothness is developed. To solve this problem, the mathematical model of relative oscillations of the sprung part is constructed, provided that they are carried out in the vertical plane. The latter represents the system of two nonlinear differential equations describing the relative displacement of the center of mass of the sprung part and the angle of rotation of the latter around the transverse axis passing through the center of mass of the specified part. To construct the approximate analytical solution of this equations system, and thus to describe the main parameters that determine the relative position of the sprung part under reasonable assumptions, the method of normal oscillations of nonlinear systems with concentrated masses is used. This made it possible to obtain the system of ordinary differential equations of the first order that describe the amplitude–frequency characteristics of the sprung part vibrations. Due to the analysis of the latter it is determined that at a certain ratio between the parameters describing the power characteristics of the suspension system, it can perform isochronous vertical and longitudinal–angular oscillations, and thus it is possible to achieve maximum comfort in transporting passengers or dangerous cargo over rough terrain. The main obtained results can be used to create the software product for adaptive suspension, and their validity is confirmed by: a) in passing to the limit, obtaining results known from literary sources; b) generalization, based on the use of periodic Ateb-functions, of widely tested analytical methods for constructing solutions of differential equations with strong nonlinearity.

Keywords: *nonlinear oscillations; adaptive suspension; method of normal oscillations; amplitude; frequency.*

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1. Introduction

To ensure the proper comfort of transporting people and special cargo over rough terrain, modern wheeled vehicles increasingly use special and adaptive suspension systems [1–4]. It has the property of "adjusting" the parameters determining the power characteristics of the suspension system depending on the movement speed, road unevenness, and therefore the oscillations caused by the latter. This primarily concerns the ensuring of the movement smoothness, controllability, movement stability, etc. [5–10]. In order to create the software product for controlling the determining power parameters of such suspension system, a priori information about the response of the sprung part to the changes in various types of motion disturbances caused by external factors and changes in the speed of the wheeled vehicle remains a problem. The latter, within the limits of one or another physical and corresponding mathematical models of the dynamics of the wheeled vehicle, can be obtained on the basis of numerical experiment or in some cases by analytical method. Generally speaking, the numerical experiment for many cases has limited resource, because the mathematical models adequate to the selected physical ones represent complex systems of nonlinear differential equations for which only it is possible to obtain approximate analytical dependencies, which could be the basis for creating the software product for controlling power parameters suspension system and kinematic parameters of wheeled vehicle movement only in some cases. Obtaining more general analytical dependencies for assessing the impact of the suspension system with non-conservative power characteristic, which would serve as the basis for creating the software product of adaptive suspension systems, is the subject of this paper, hence its relevance.

2. Analysis of literature data and statement of the problem

Wheeled vehicles operating at high speeds and in difficult conditions, such as driving on rough terrain, should meet increased performance characteristics. They relate not only to the engine, transmission, and other components or systems, but primarily to the suspension system [5, 10–14]. The latter must ensure proper movement smoothness and protect people, cargo, and equipment from overloads (excessive oscillations [15]). Suspension systems of such vehicles with linear or close to them laws of change in the elastic force of shock absorbers and the resistance force of damping devices do not only protect against significant overloads (including instantaneous ones), but also result in significant tiredness of the driver or people during long-distance transportation. As shown by experimental [12] and theoretical investigations [16–18], the elastic force of the shock absorbers of such wheeled vehicles acting on the sprung mass should meet certain conditions, namely, it must be small for insignificant deformations of the shock absorbers and increase significantly with their considerable values. In addition, the oscillation frequency of the sprung part should vary within a certain range. Such requirements are partially met by the suspension with nonlinear relationship between the restoring force and strain [19–22] and the adaptive suspension [3, 4], which is used for passenger cars by leading automobile manufacturers [7]. However, analytical investigations of the dynamics of the sprung part of the wheeled vehicle with these types of suspensions have not been properly developed due to purely technical and mathematical problems. Even some investigations concerning simplified calculation and, consequently, mathematical models have shown the fundamental difference in their dynamics compared to the dynamics of the sprung part according to the linear characteristics of the suspension system. This primarily concerns the dependence of the frequency of natural oscillations on the amplitude, and thus the conditions for the existence of resonant oscillations, the dependence of the resonant amplitude on the nonlinear power characteristic, speed, etc. It is a problematic task to obtain such generalized results based on numerical analysis of the corresponding nonlinear mathematical models of the dynamics of the sprung part of the wheeled vehicle. The development of the methodology for analyzing the influence of nonlinear power characteristics of the suspension system on the dynamics of the suspended mass, which would be the basis for creating new or modernizing existing suspensions, in particular adaptive ones, is the subject of this paper. Such way, in our opinion, is the most reasonable and resource (material) efficient.

3. Investigation objectives

Determination of the basic parameters of promising wheeled vehicle suspensions cannot be carried out without conducting thorough theoretical investigations devoted to the creation of new or refinement of existing mathematical models of their dynamics, and thus the adaptation of existing or creation of new methods for theoretical research of the developed models. First of all, it is necessary to solve the problem, as a rule, of analytical research of the influence of basic parameters describing nonlinear characteristic of the suspension system depending on the deformation, and in many cases the deformation rate of its elastic elements. After all, the experience of operating samples of wheeled vehicles that use suspension systems with linear or even linear with discrete law of change in the stiffness of elastic shock absorbers under conditions of maximum (proper) ensuring of smooth movement while driving along the road with irregularities reduces the service life of individual components. At the same time, almost all theoretical investigations of the dynamics of the wheeled vehicle are carried out with linear power characteristics of the suspension system. Therefore, the investigations of the wheeled vehicle with nonlinear power characteristics of the suspension system, which is operated in difficult conditions, should be carried out in such a way as to obtain analytical dependencies from the ratios of the dynamics of the suspended part that would describe the main parameters of the relative oscillations of the suspended part and at the same time be the basis for creating the software product for adaptive suspensions of the wheeled vehicle. It is this task that is partially solved in this paper, using the example of longitudinal-angular vibrations of the wheeled vehicle's suspension part.

4. Investigation results

The longitudinal and vertical vibrations of the sprung part of a wheeled vehicle are important when assessing such operational characteristics as movement smoothness, controllability, movement stability, etc. As noted above, suspension systems with nonlinear characteristics of elastic shock absorbers and adaptive suspension can ensure their proper functioning when the wheeled vehicle is moving along the road with bumps and rough terrain. For the latter, an important problem is the creation of basic principles for creating the software product, which is the subject of this paper. Thus, for the wheeled vehicle, we consider longitudinal–angular vibrations of the sprung part, provided that the restoring force of the shock absorbers is described by the following dependencies

$$F_i(\Delta_i, \dot{\Delta}_i) = \left(\alpha_i + \beta_i (\dot{\Delta}_i)^{\nu_1}\right) (\Delta_i)^{\nu_2+1}, \tag{1}$$

where Δ_i and $\dot{\Delta}_i$ are, respectively, the deformation and deformation rate of the front ($i = 1$) and rear ($i = 2$) shock absorbers of the front and rear axles; α_i, β_i, ν_1, ν_2 are the constants selected from the conditions of wheeled vehicle comfort. For purely mathematical reasons, ν_1, ν_2, should ensure the oscillatory process of the sprung part, so it can take the value $\nu_1 + 1 = (2m + 1)/(2n + 1)$, $\nu_2 + 1 = (2p + 1)/(2q + 1)$ $(m, n, p, q = 0, 1, 2, \ldots)$. As for the resistant forces of the damping devices, they, as in most cases of linear or nonlinear systems [10], depend on the deformation rate of the elastic elements of suspension system and are described by nonlinear functions of this value - $R_{\text{ioop}} = \chi_i \dot{\Delta}_i^{2s+1}$, ($\chi_i$ and s are constant), and their maximum values are small compared to the maximum value of the restoring forces. The task is to derive analytical dependencies based on the longitudinal–angular oscillations of the sprung part that describe the influence of the entire set of the above mentioned parameters on the amplitude–frequency response of the oscillations of the sprung mass, and which would simultaneously serve as the basis for creating the software product for controlling the power parameters of the suspension system.

In order to solve this problem, a plane two-mass system with the unsprung and sprung part is chosen as the calculation model. The sprung part oscillates in the vertical plane relative to the unsupported part. Therefore, the relative position is definitely determined by the position of the center of mass of this part and the angle of rotation of it around the transverse axis passing through the specified point (angle φ). The vertical position of the center of sprung part mass is calculated from the position of static equilibrium of the specified part. Therefore, the deformations of the elastic elements of the suspension system for arbitrary position of the sprung part are determined by the dependencies $\Delta_1 = l_1\varphi(t) + z(t) - \Delta_{\text{st}}$, $\Delta_2 = l_2\varphi(t) + \Delta_{\text{st}} - z(t)$, l_1, l_2 are parameters determining the position of the center of sprung part mass, Δ_{st} is static deformation of elastic shock absorbers, which, taking into account (1), takes the value $\Delta_{\text{st}} = \sqrt[\nu_2+1]{\frac{Q}{\alpha_1+\alpha_2}}$, Q is weight of the sprung part. Accordingly, the deformation rates of the contact points of shock absorbers or damping devices with the sprung part are determined.

Remark 1. The elastic properties of tires as road irregularities are neglected in this work. Such problems can be the subject of separate studies.

All of the above mentioned makes it possible to write the differential equations of relative oscillations of sprung part in the following form

$$
\begin{aligned}
I_0\ddot{\varphi} &= -l_1\big(\big(\alpha_1+\beta_1(a\dot{\varphi}+\dot{z})^{\nu_1}\big)\big(l_1\varphi-\Delta_{\mathrm{st}}+z\big)^{\nu_2+1}\big)\\
&\qquad - l_2\big(\big(\alpha_2+\beta_2(l_1\dot{\varphi})^{\nu_1}-\dot{z}\big)\big(l_2\varphi+\Delta_{\mathrm{st}}-z\big)^{\nu_2+1}\big)+M_0(\dot{\varphi},\dot{z}),\\
\frac{Q}{g}\ddot{z} &= Q-\big(\big(\alpha_1+\beta_1(l_1\dot{\varphi}+\dot{z})^{\nu_1}\big)\big(l_1\varphi-\Delta_{\mathrm{st}}+z\big)^{\nu_2+1}\big)\\
&\qquad + \big(\big(\alpha_2+\beta_2(l_2\dot{\varphi})^{\nu_1}-\dot{z}\big)\big(l_2\varphi+\Delta_{\mathrm{st}}-z\big)^{\nu_2+1}\big)+R(\dot{\varphi},\dot{z}),
\end{aligned}
\tag{2}
$$

where I_0 is inertia moment of the sprung part relative to the transverse axis passing through the center of mass of the specified part, $R(\dot{\varphi},\dot{z})$ and $M_0(\dot{\varphi},\dot{z})$ are, respectively, the main vector and the resistance force moment relative to the above mentioned axis.

It is necessary to determine the influence of the main parameters characterizing the restoring forces of elastic shock absorbers on the amplitude–frequency characteristic of the sprung part oscillations, and then to provide practical recommendations for their selection so that the suspension system meets the comfortable transportation of goods and people over rough terrain and along the road with irregularities. For this purpose, let us go on to the approximate construction of the analytical solution of the nonlinear system of differential equations (2). In order to do this, we will represent accurately the values of higher order of smallness in the following form

$$
\begin{aligned}
&I_0\ddot{\varphi}+l_1\big(\big(\beta_1(l_1\dot{\varphi}+\dot{z})^{\nu_1}\big)(l_1\varphi+z)^{\nu_2+1}\big)-l_2\big(\big(\beta_2(l_2\varphi-\dot{z})^{\nu_1}\big)\big(l_2\varphi-z\big)^{\nu_2+1}\big)=M_0(\dot{\varphi},\dot{z}),\\
&\frac{Q}{g}\ddot{z}+\big(\big(\beta_1(l_1\dot{\varphi}+\dot{z})^{\nu_1}\big)(l_1\varphi+z)^{\nu_2+1}\big)+\big(\big(\beta_2(l_2\dot{\varphi})^{\nu_1}-\dot{z}\big)(l_2\varphi-z)^{\nu_2+1}\big)=R_0(\dot{\varphi},\dot{z}).
\end{aligned}
\tag{3}
$$

Based on the conditions satisfied by the resistance forces of the damping devices, it can be stated that the maximum values of the right-hand parts of the differential equations (3) are small values compared to the maximum values of the terms of the left-hand parts of the differential equations. This is the basis for applying the general ideas of perturbation methods while constructing the solution of the equation system (3). According to them, first of all, it is necessary to find the solution of the generating system, that is, the system of nonlinear differential equations

$$
\begin{aligned}
&I_0\ddot{\varphi}+l_1\big(\big(\beta_1(l_1\dot{\varphi}+\dot{z})^{\nu_1}\big)(l_1\varphi+z)^{\nu_2+1}\big)-l_2\big(\big(\beta_2(l_2\dot{\varphi}-\dot{z})^{\nu_1}\big)(l_2\varphi-z)^{\nu_2+1}\big)=0,\\
&\frac{Q}{g}\ddot{z}+\big(\big(\beta_1(l_1\dot{\varphi}+\dot{z})^{\nu_1}\big)(l_1\varphi+z)^{\nu_2+1}\big)+\big(\big(\beta_2(l_2\dot{\varphi})^{\nu_1}-\dot{z}\big)(l_2\varphi-z)^{\nu_2+1}\big)=0.
\end{aligned}
\tag{4}
$$

The solution of the nonlinear system of differential equations (4) will be sought using the method of normal oscillations [23–28]. According to its main idea, the unknown functions $\varphi(t)$ and $z(t)$ must be connected by the relation $z(t)=\lambda\varphi(t)$. The problem is to find such value λ and function for $\varphi(t)$ of which the relation (4) is true. To find them, we get the following ratio

$$
\begin{aligned}
I_0\ddot{\varphi} &= -l_1\left(\big(\beta_1(l_1\dot{\varphi}+\lambda\dot{\varphi})^{\nu_1}\big)(l_1\varphi+\lambda\varphi)^{\nu_2+1}\right)-l_2\left(\big(\beta_2(l_2\dot{\varphi}-\lambda\dot{\varphi})^{\nu_1}\big)(l_2\varphi-\lambda\varphi)^{\nu_2+1}\right),\\
\frac{Q}{g}\lambda\ddot{\varphi} &= -\left(\big(\beta_1(l_1\dot{\varphi}+\lambda\dot{\varphi})^{\nu_1}\big)(l_1\varphi+\lambda\varphi)^{\nu_2+1}\right)+\left(\big(\beta_2(l_2\dot{\varphi})^{\nu_1}-\lambda\dot{\varphi}\big)(l_2\varphi-\lambda\varphi)^{\nu_2+1}\right).
\end{aligned}
\tag{5}
$$

The last dependence at $l_1=l_2=l$, $\beta_1=\beta_2=\beta$ makes it possible to find the following approximate parameter values λ: $\lambda_1=0$, $\lambda_2=\sqrt{\frac{i_x^2(\nu_1+\nu_2+1)-2l^2}{(\nu_1+\nu_2)(\nu_1+\nu_2+1)}}$, $\lambda_3=-\sqrt{\frac{i_x^2(\nu_1+\nu_2+1)-2l^2}{(\nu_1+\nu_2)(\nu_1+\nu_2+1)}}$, where i_x is radius of inertia of the sprung part relative to the transverse axis. Note that the first value of the parameter λ ($\lambda_1=0$) correspond to the longitudinal angular oscillations of the sprung part and were considered in [23]. Therefore, below we will consider more complex cases of the sprung part oscillations — vertical and longitudinal angular vibrations of the sprung part of the wheeled vehicle. Non-linear differential equations correspond to them without taking into account the resistance forces of the damping devices

$$
\begin{aligned}
&I_0\ddot{\varphi}+l\beta\rho_i(\dot{\varphi})^{\nu_1}\varphi^{\nu_2+1}=0,\\
&\frac{Q}{g}\ddot{z}+\beta\bar{\rho}_i(\dot{z})^{\nu_1}z^{\nu_2+1}=0,
\end{aligned}
\tag{6}
$$

where $\rho_i=\big((l+\lambda_i)^{\nu_1+\nu_2+1}-(l-\lambda_i)^{\nu_1+\nu_2+1}\big)$, $\bar{\rho}_i=\big(\big(\frac{l}{\lambda_i}+1\big)^{\nu_1+\nu_2+1}+\big(\frac{l}{\lambda_i}-1\big)^{\nu_1+\nu_2+1}\big)$.

The structure of these equations is the same. In addition, from the methodologies for obtaining them and the condition of normal sprung part oscillations, their solutions are expressed [26] through periodic Ateb functions [29, 30] by the following dependence

$$\varphi = aca\left(\nu_2+1,(1-\nu_1)^{-1},\omega(a)t+\theta\right), \quad z = a\lambda_i ca\left(\nu_2+1,(1-\nu_1)^{-1},\omega(a)t+\theta\right), \tag{7}$$

where a is the amplitude of longitudinal-angular vibrations, and

$$\omega(a) = \frac{\nu_2+2}{2}\left(\frac{(2-\nu_1)\beta l}{I_0(1-\nu_1)(\nu_2+2)}\rho_i\right)^{\frac{1}{2-\nu_1}} a^{\frac{\nu_1+\nu_2}{2-\nu_1}}. \tag{8}$$

The dependence that takes into account the static deformation of elastic shock absorbers is more convenient for the practical determination of the frequency of natural oscillations. At $\alpha_1 = \alpha_2$ it is as follows

$$\omega(a) = \frac{\nu_2+2}{2}\left(\frac{g(2-\nu_1)\beta l}{2\alpha(\Delta_{\text{ct}})^{\nu_2+1}(i_x)^2(1-\nu_1)(\nu_2+2)}\rho_i\right)^{\frac{1}{2-\nu_1}} a^{\frac{\nu_1+\nu_2}{2-\nu_1}}. \tag{9}$$

The oscillation frequencies of the sprung part in hertz $f = \frac{\omega(a)}{2\Pi}$ (P is the half-period of the used Ateb functions, i.e. $\Pi = \Gamma\left(\frac{1-\nu_1}{2-\nu_1}\right)\Gamma\left(\frac{1}{\nu_2+2}\right)\Gamma^{-1}\left(\frac{1-\nu_1}{2-\nu_1}+\frac{1}{\nu_2+2}\right)$) from the static deformation at the following values of the amplitude of longitudinal–angular vibrations 1) $a = 0.05$ (red), 2) $a = 0.25$ (blue), 3) $a = 0.4575$ (green), and different values of the parameters ν_1, ν_2 and $\alpha = 800000$, $\beta = 500000$, $l = 2$ in accordance with dependence (9), for different values of the parameters characterizing wheeled vehicle suspension systems are shown below in Figure 1.

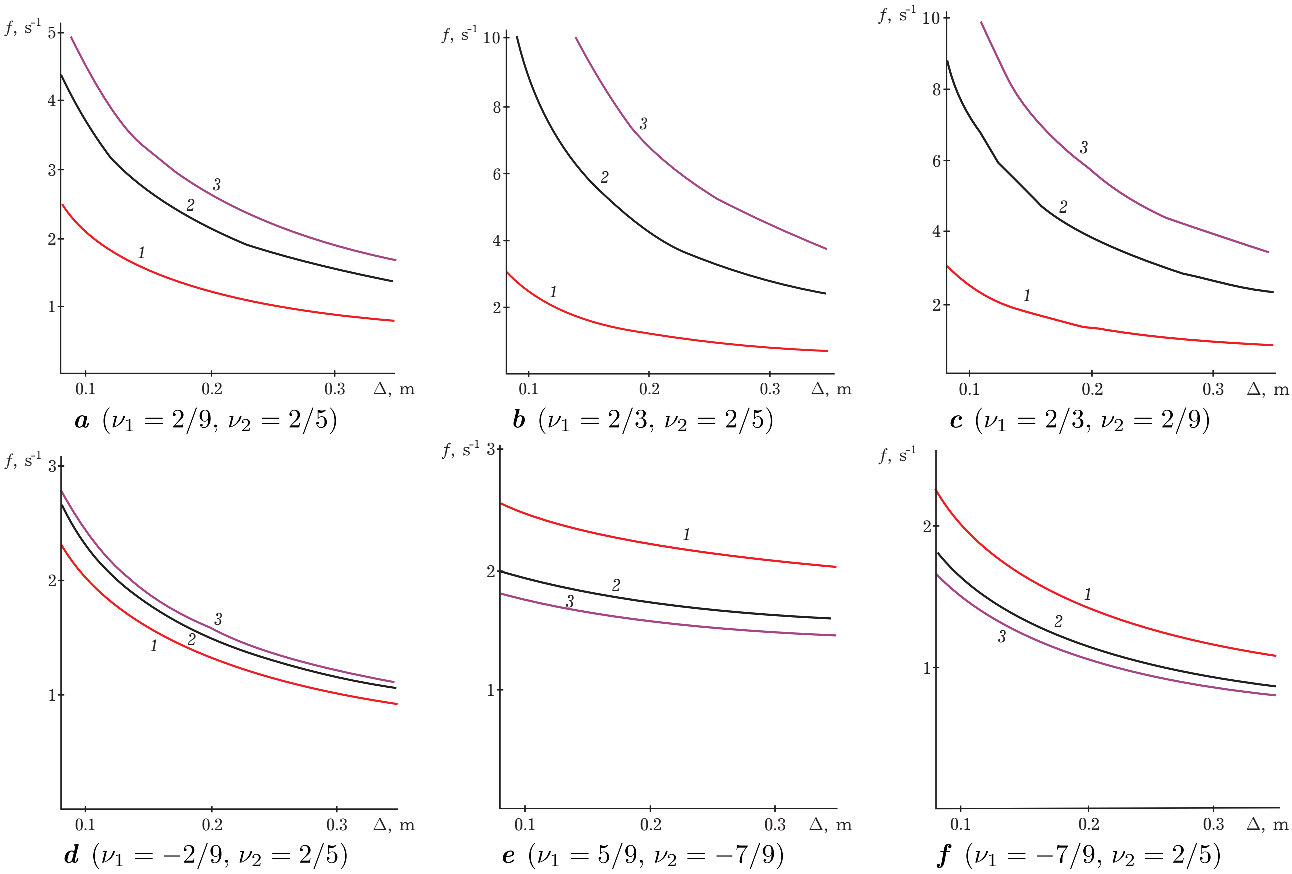

Fig. 1. Dependence of the natural oscillation frequency of the sprung part on the static strain for different values of the oscillation amplitudes and parameters ν_1, ν_2.

The obtained dependences and the graphical dependences built on their basis show that the natural frequency of the normal (longitudinal–angular and vertical) for larger values of static deformation takes on smaller value. As for the influence of the oscillations amplitude on frequency, to a greater extent for a wide range of its change, ergonomic conditions are satisfied by the adaptive suspension with parameter value close to $\nu_1 = 2/9$, $\nu_2 = 2/5$; $\nu_1 = -2/9$, $\nu_2 = 2/5$; $\nu_1 = 5/9$, $\nu_2 = -7/9$. In addition, for the first two cases, the higher value of the oscillation amplitude corresponds to the higher value of the natural frequency, and for the third case, the lower value corresponds to the lower value.

At the same time, the influence of the damping force of the damping devices on the oscillations of the sprung part is manifested in the damping of normal oscillations and, provided that the damping force is proportional to the speed with proportionality coefficient γ, for the first approximation, the amplitude of oscillations is described by the following dependence

$$\dot{a}(t) = -\gamma \frac{2-\nu_1}{\Pi} \frac{2a(t)}{\nu_2+2} \frac{2\Gamma\left(\frac{1-2\nu_1}{2-\nu_1}\right)\Gamma\left(\frac{1}{\nu_2+2}\right)}{\Gamma\left(\frac{1}{\nu_2+2}+\frac{1-\nu_1}{2-\nu_1}\right)}. \quad (10)$$

As for the qualitative picture of oscillation damping, it does not differ significantly, for example, from the results obtained in [23] for nonlinear vibrations of the sprung part.

5. Discussion of the investigation results

The results obtained in this paper can be used in the early stages of suspensions designing, in particular, in the creation of adaptive or semi-adaptive software products. They make it possible to:

— determine the main parameters of the suspension system of a wheeled vehicle, taking into account the conditions of its operation;
— be the basis for determining such operational characteristics as controllability, stability of movement along curved sections of the road;
— form traffic routes and determine the maximum possible traffic speed.

6. Conclusions

The obtained analytical and graphical dependences built on their basis show that the natural frequency of longitudinal and angular oscillations of the sprung part of a wheeled vehicle with controlled suspension system depends significantly not only on the static deformation of elastic shock absorbers, but also on the amplitude of oscillations and the relationship between the parameters ν_1, ν_2. It is these parameters that indicate the deviation of elastic properties of the suspension system from the linear law. As for the dependence of the frequency of the sprung mass oscillations on the static deformation, for controlled suspensions, for the larger value of its magnitude, the natural frequency is lower. In addition, the ergonomic conditions of operation are more satisfied by suspensions with the following values of nonlinearity parameters $\nu_1 > 0$, $\nu_2 > 0$ for small amplitudes of oscillations and suspension deformation of elastic shock absorbers which $0.2\,\mathrm{m} < \Delta_{\mathrm{st}} < 0.35\,\mathrm{m}$ and $-1 < \nu_1 < 0$, $\nu_2 > 0$ for large amplitudes of oscillations. If we consider the dynamic process of the sprung mass as continuous (damping oscillations after hitting a bump in the road) in a wide range of changes in amplitudes of oscillations, then the most favorable from the ergonomic point of view [14] is the controlled suspension with the following characteristics: $0.2\,\mathrm{m} < \Delta_{\mathrm{st}} < 0.3\,\mathrm{m}$ at $0 < \nu_1 < 2/5$ and $0 < \nu_2 < 8/7$.

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Метод нормальних коливань та обгрунтування вибору параметрів деяких нелінійних систем з двома ступенями вільності

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На прикладі плоскої моделі коливань колісного транспортного засобу із адаптивною силовою характеристикою системи підвіски розроблено методику вибору основних її параметрів, які б максимально забезпечували плавність ходу. Для вирішення поставленої проблеми побудовано математичну модель відносних коливань підресореної частини за умови, що вони здійснюються у вертикальній площині. Остання являє собою систему двох нелінійних диференціальних рівнянь, які описують відносне переміщення центру мас підресореної частини та кут повороту останньої навколо поперечної осі, яка проходить через центр мас вказаної частини. Для побудови наближеного аналітичного розв'язку вказаної системи рівнянь, а відтак описання основних параметрів, які визначають відносне положення підресореної частини, за обгрунтованих припущень використано метод нормальних коливань нелінійних систем із зосередженими масами. Це дозволило отримати систему звичайних диференціальних рівнянь першого порядку, які описують амплітудно-частотну характеристику коливань підресореної частини. Аналізом останньої, зокрема, встановлено, що: за певного співвідношення між параметрами, які описують силові характеристики системи підвіски, вона може здійснювати ізохронні вертикальні та поздовжньо-кутові коливання, а відтак можна досягнути максимальної комфортності перевезення пасажирів чи небезпечних вантажів пересіченою місцевістю. Справедливість отриманих основних результатів підтверджується: а) отриманням у граничному випадку результатів відомих із літературних джерел; б) узагальненням на базі використання періодичних Ateb-функцій широко апробованих аналітичних методів побудови розв'язків диференціальних рівнянь із сильною нелінійністю. Отримані результати можуть бути використані для створення програмного продукту адаптивної підвіски.

Ключові слова: *нелінійні коливання; адаптивна підвіска; метод нормальних коливань; амплітуда; частота.*