LOCAL FITS OF SIGNALS WITH ASYMPTOTIC BRANCHES

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ABSTRACT. The method of "asymptotic parabola" fit is described, its analytical properties are discussed in comparison with other types of fits. The method is effective for the (possibly highly asymmetric) signals with practically linear ascending and descending branches connected by relatively short transitions at maximum or minimum, e.g. for the brightness variations of pulsating or eclipsing variables or for the phase variations in stars with abrupt period changes. The method is illustrated by an application to the Mira–type star U Her.

Key words: Data Reduction – Variables: Pulsating: U Her

Introduction

The individual cycles of variability of many stars show phase intervals of abrupt changes and/or undergo significant changes of the shape (e.g. Kholopov et al. 1985). To study these objects, several methods are usually applied - the fitting by ordinary and trigonometric polynomials, splines, (multi–) Gaussian functions etc. A number of "running" approximations may also be applied, the statistical properties of which for arbitrary basic functions and additional weight functions were recently discussed by Andronov (1997).

Andronov (1994) described some algorithms used in his programs for time series analysis. In this paper we describe a supplementary method of smoothing which we call the method of "asymptotic parabolae" (AP). Its error estimates are sometimes better than the polynomial or "running parabola" fits. The main idea is to split the observational interval near extremum into the parts with linear ("asymptotic") branches which are connected by a transition curve. This also will allow to determine the characteristic time of a transition from one linear branch to another.

To study abrupt "switches" of the outburst cycle length of some dwarf novae between two preferred values, Andronov and Shakun (1990) have used hyperbolic functions. However, Andronov (1995) noted that more simple parabolic connection between the lines takes less computational time and often corresponds to better accuracy estimate of the position of the extremum. A corresponding program was developed with an algorithm described below in detail.

The "Asymptotic Parabola" Fit

Assuming the signal is represented by the observational values y_k obtained at times t_k, $k = 1...n$, one may obtain the coefficients of the fit

$$y_c(t) = \sum_{\alpha=1}^{m} C_\alpha f_\alpha(t) \qquad (1)$$

from the system of normal equations

$$\sum_{\alpha=1}^{m} A_{\alpha\beta} C_\alpha = \sum_{k=1}^{n} y_k\, f_\beta(t_k). \qquad (2)$$

Here $y_c(t)$ is a value of the smoothing function at argument t, $f_\alpha(t)$ are basic functions ($\alpha = 1...m$), and

$$A_{\alpha\beta} = \sum_{k=1}^{n} f_\alpha(t_k)\, f_\beta(t_k) \qquad (3)$$

is the matrix of normal equations (e.g. Whittaker and Robinson, 1926). We repeat these well–known expressions to apply them to the proposed basic functions.

Among the infinite number of functions $f_\alpha(t)$ we have chosen the basic functions which may be written in a form:

$$f_1(t) = 1, \quad f_2(t) = F(z), f_3(t) = -F(z). \qquad (4)$$

Here we used the transformation $z = (t - t_0)/\Delta t$, where t_0 is some characteristic value of the argument t, and Δt is a characteristic scale of the argument (or $2\Delta t$ is the "effective duration" of the transition).

The smoothing function remains the same, if one will choose the a new set of basic functions which are *linear* combinations of the initial ones. This is the case e.g. for the ordinary polynomial $f_\alpha(t) = t^{\alpha-1}$ or trigonometric polynomial fits which are not dependent on t_0. Moreover, the ordinary polynomial fit does not depend also on Δt, contrary to a trigonometric polynomial.

In the case of non–linear transformation of the basic functions one may also include the corresponding "non–linear" parameters to the list of unknowns and make the optimization according to these parameters.

As an additional assumption, we introduce the function $F(z)$ with asymptotes for large positive z : $F(-z) = 0, F(+z) = z$, with corresponding asymptotic expressions for $x_c(t) = C_3F(-z)$ and $C_2F(+z)$. Obviously, such fits with asymptotes depend both on t_0 and Δt which must be determined as the values minimizing the r.m.s. deviation from the fit. The algorithm is the usual one: the preliminary values are determined by choosing the minimum value at a grid of trial values and then corrected by using some iterations.

The asymptotic functions $F(z) = 0$ $(z \leq -1)$ and $F(z) = z$ $(z \geq +1)$ may be connected by a parabola $F(z) = (1+z)^2/4$ $(-1 \leq z \leq +1)$ which satisfies the conditions of continuity of the smoothing function and its first derivative at all arguments (including the border points).

For illustration, the fits are computed for one interval of observations of the Mira–type star U Her from the AFOEV database (Schweitzer 1996).

For this "asymptotic parabola" (AP) fit one may choose the borders in such a way that they will be equal to $t_{*1} = t_0 - \Delta t$ and $t_{*2} = t_0 + \Delta t$. This will correspond to the border values $z_1 = -1$ and $z_2 = +1$.

One may note that the AP fit may be represented by more usual functions. For example, if $t_{*1} \leq t_1$ and $t_{*2} \geq t_n$ (where t_1 and t_n are the smallest and largest arguments of the signal), than the data will be fitted by an ordinary parabola. If $t_{*1} = t_{*2} = t_0$, then the duration $2\Delta t$ of transition is equal to zero, and the fit is an "angle–like" broken line. If $t_{*1} \geq t_n$ or $t_{*2} \leq t_1$, one will obtain a single line without any transition to another asymptote. "One–line" + parabola fits correspond to $t_{*1} \leq t_1$ and $t_1 < t_{*2} < t_n$ or to $t_1 < t_{*1} < t_n$ and $t_{*2} \geq t_n$. Only the case $t_1 < t_{*1} < t_{*2} < t_n$ corresponds to the AP with two asymptotes. These remarks are obvious but all these cases are to be taken into account in the algorithm while computing the differential corrections.

In the computer program we have checked also the condition $t_{*1} < t_{*2}$. If needed, these values were swapped to obtain the ascending order.

If $t_{*1} = t_{*2} = t_0$, then we decreased the number of unknowns leaving only t_0. In this case the parabolic connection of the asymptotes is equal to zero, and one should use the modified function $F(z) = 0$, if $z \leq 0$ and $F(z) = z$, if $z \geq 0$, where in this case $z = t - t_0$ without dividing by Δt.

Determination of the "Non–Linear" Parameters

The root mean squared error $\sigma[y_c^{[s]}(t)]$ of the smoothing function ($s = 0$) and its derivatives of the s^{th} order in respect to the parameter t

$$\sigma^2[y_c^{[s]}(t)] = \sigma_0^2 \sum_{\alpha\beta=1}^{m} A_{\alpha\beta}^{-1} f_\alpha^{[s]}(t) f_\beta^{[s]}(t), \qquad (5)$$

where σ_0 is the error estimate corresponding to an "unit weight" (e.g. Whittaker and Robinson 1926) and may be computed according to the equation

$$\sigma_0^2 = \frac{1}{n-m} \sum_{k=1}^{n} (y_k - y_c(t_k))^2, \qquad (6)$$

where $A_{\alpha\beta}^{-1}$– is the matrix, inverse to $A_{\alpha\beta}$.

The values of σ_0 are dependent on the 'non–linear' parameters – the pairs t_0 and Δt or t_{*1} and t_{*2}. From the computational point of view, the grid should be computed for different data sets on similar number of points. Thus one may recommend to use the scaled arguments, e.g. $x = (t-t_1)/(t_n-t_1)$. In our program, the function $\sigma(x_1, x_2)$ is computed for $0 \leq x_1 \leq x_2 - s_x$, $s_x \leq x_2 \leq 1$ with an adopted step $s_x = 0.05$. Here x_1 and x_2 correspond to the times t equal to t_{*1} and t_{*2}. This dependence is shown in Fig.1. To compute the surface, we have used also the values $x_1 \geq x_2$ swapping them for computation. Thus, in our definition, $\sigma(x_2, x_1) = \sigma(x_1, x_2)$ and the surface is symmetrical in respect to the line $x_1 = x_2$. One may see a wide "valley" with a single minimum in a region $x_1 \leq x_2$. The contour map of the smaller region near the minimum $0 \leq x_1 \leq 0.4$ and $0 \leq x_2 \leq 0.4$ computed with a smaller step $s_x = 0.01$ is shown in Fig.2.

The lines of constant σ are elliptic only in the vicinities of the minimum with a major semi-axis directed nearly along the line t_0 =const and a minor semi–axis along the line Δt =const. With increasing σ, the lines $\sigma(x_1, x_2)$ =const are strongly deformed having a joint point at $x_1 = x_2 = x_{m0}$. However, this value corresponds to the minimum of $\sigma(x_0 - \Delta t, x_0 + \Delta t)$ for the fixed value of $\Delta t = 0$, but the true minimum may be found by determination of the optimal value of Δt.

For symmetrical input signal (same observational values for x and $1-x$), one may expect that the function $\sigma(x_1, x_2)$ will be symmetrical not only in respect to the line $x_1 - x_2 = 0$, but in respect to the line $x_1 + x_2 = 1$ as well. In this case $x_0 = x_{m0} = 0.5$ and one has to find Δt only, i.e. the minimum will occur on the line $x_1 + x_2 = 1$. For asymmetric signals the function $\sigma(x_0 - \Delta t, x_0 + \Delta t)$ will reach its minimum along the line Δt =const at the values of x_0 being dependent on Δt. The possible consequence of this curvature is that the minimization of $\sigma(x_0 - \Delta t, x_0 + \Delta t)$ may be obtained by preliminary determination of x_{m0} for $\Delta t = 0$, then minimization in respect to Δt for the fixed x_{m0}, then for the fixed Δt in respect to x_0 etc. This method of "cyclic" optimization was described e.g. by Korn and Korn (1970, p. 576).

Differential Corrections

The differential corrections δC_α, $\delta\nu$, δz_0, at each iteration may be determined by using the system of "conditional" equations

$$\sum_{\alpha=1}^{m} f_\alpha(t_k)\,\delta C_\alpha + \frac{\partial x_c}{\partial \nu}\,\delta\nu + \frac{\partial x_c}{\partial z_0}\,\delta z_0 = x_k - x_c(t_k). \quad (7)$$

Here we introduced new variables $\nu = \Delta t^{-1}$ and $z_0 = \nu t_0$, thus $z = \nu t - z_0$. One may note that the partial derivative of the smoothing function in respect to the "non–linear" parameter μ is

$$\frac{\partial x_c(t)}{\partial \mu} = \sum_{\alpha=1}^{m} C_\alpha \frac{\partial f_\alpha(t)}{\partial \mu} = \sum_{\alpha=1}^{m} C_\alpha \frac{\partial f_\alpha(t)}{\partial t}\frac{\partial t}{\partial \mu} \quad (8)$$

After some iterations one may determine $m+2$ parameters C_α, ν, δz_0 and corresponding error estimates.

Andronov and Shakun (1990) have used an analytic function

$$F(z) = \frac{1}{2}(z + \ln(e^z + e^{-z})) = z + \frac{1}{2}\ln(1+e^{-2z}) \quad (9)$$

which is differentiable infinite number of times. The corresponding derivative of the smoothing function is generally equal to

$$\frac{\partial x_c(t)}{\partial t} = \sum_{\alpha=1}^{m} C_\alpha \frac{\partial f_\alpha(t)}{\partial t} \quad (10)$$

and in our case

$$\frac{\partial x_c(t)}{\partial t} = \left(C_2 \frac{dF(z)}{dz} - C_3 \frac{dF(-z)}{d(-z)} \right)\nu. \quad (11)$$

For the "asymptotic" fits, $dF(z)/dz \rightarrow 0$ for large negative z and $\rightarrow 1$ for large positive z, thus $\partial x_c(t)/\partial t \rightarrow -C_3$ and $\rightarrow C_2$. For the function (9) $dF(z)/dz = \frac{1}{2}(1+\tanh z) = 1/(1+e^{-2z})$, and the fit may be called a "hyperbolic tangent" (HT) one. For practical purposes one may not compute the fit precisely for very large positive or negative z making simplification by neglecting the term $e^{-|z|}$ when compared with unity. Thus one may conclude that the whole interval of arguments is splitted into three parts - large negative, intermediate and large positive values. However, the borders are not single functions on Δt and t_0.

The first derivative (which is needed for differential corrections) is also very simple:

$$\frac{dF(z)}{dz} = \begin{cases} 0 & \text{if } z \le -1 \\ (1+z)/2 & \text{if } -1 \le z \le +1 \\ 1 & \text{if } +1 \le z \end{cases} \quad (12)$$

However, the apparent simplicity of this derivative does not cause very simple computation of the differential corrections, and there are some problems which have been taken into account in the algorithm for application of this method for the time series analysis.

At first, we have introduced the independent variables t_0 (i.e. the argument, at which the asymptotic lines cross each other) and Δt (the complete duration of the transition between the asymptotes). The corresponding derivatives $\partial z/\partial t_0 = \Delta t^{-1}$ and $\partial z/\partial \Delta t = -(t-t_0)\Delta t^{-2}$ are dependent on Δt and tend to infinity while $\Delta t \rightarrow 0$. However, one may not expect any singularity when choosing the fits with $\Delta t \rightarrow 0$, thus this problem may be solved. Slightly better solution is to use the variables ν instead of Δt, thus $z = \nu(t-t_0)$ and the derivatives are equal to $\partial z/\partial t_0 = -\nu$ and $\partial z/\partial \nu = t - t_0$. Introducing the parameter z_0, i.e. $z = \nu t - z_0$ one may simplify the derivatives.

Moreover, the matrix of normal equations may be degenerate because for the "bad" location of x_1 and x_2, the values of the fit at arguments of data may not depend on x_1 and/or x_2 (e.g. when no data are present in the interval of transition). Thus practically there must be an automatic checking the degeneracy of the matrix of normal equations the number of parameters being determined. If needed, they must be changed, e.g. from x_1 and x_2 to single x_0; from two borders to one border, if another one coincides with a data limit.

Because the intersections of the surface $\sigma_O(x_1, x_2)$ by the plane $x_1 = const$ or $x_2 = const$ are parabolic only in small vicinities of the minimum (Fig. 3), sometimes the differential corrections may not converge. In this case we use the values of δx_1 and δx_1 multiplied by a parameter $\alpha \ll 1$ increase the number of iterations.

Comparison with Other Methods

Generally, the AP fit corresponds to 5 unknowns (including x_1 and x_2). Thus one has to compute the error estimates by using the matrix $A^{-1}_{\alpha\beta}$ of order 5. The corresponding dependence on t of the accuracy of the smoothing function $\sigma_c(t)$ is shown in Fig 4 and is marked as AP 5. However, the accuracy estimate of the same fit when using the matrix 3× (assuming x_1 and x_2 are fixed and have the best fit values; marked as AP 3) is much smaller than for AP 5. The corresponding error estimates $\sigma[t_e]$ of the moment of extremum differ by a factor of 3! This difference is natural taking into account a high degree of degeneracy of the matrix $A_{\alpha\beta}$. Even more drastic difference is between the AP fits with $\Delta t = 0$ which are marked as BL (broken line) with 3 and 4 parameters. For BL 3 $\sigma[t_e] = 0$, because $t_{*1} \le t_e \le t_{*2}$, and only use of the BL 4 fit allows to make an accuracy estimate.

For comparison, we have also computed the polynomial fits (P) with 3, 4 and 5 parameters (i.e. of the orders 2,3,4). The P 3 fit coincides with the AP fit with $t_{*1} = t_1$ and $t_{*2} = t_n$ which is the worst in our sample. The fit P 5 with the same number of parameters as AP 5 has similar $\sigma[m_e]$, but 2 times larger $\sigma[t_e]$. The best order of the polynomial is 3 (AP 4). The running parabola (RP) fit (Andronov, 1990) with $\Delta t = 89^{\rm d}$ has very good value σ_0, which is close to that of AP, but large $\sigma[t_e]$. The cubic parabola P 4 has the best value of $\sigma[t_e]$ which is slightly larger than that for AP, but much larger systematic deviations from the fit. The BL 4 fit is better than P 3 arguing for the "asymptotes". Similar results were obtained for other data. Thus the AP fit is optimal and may be recommended for use.

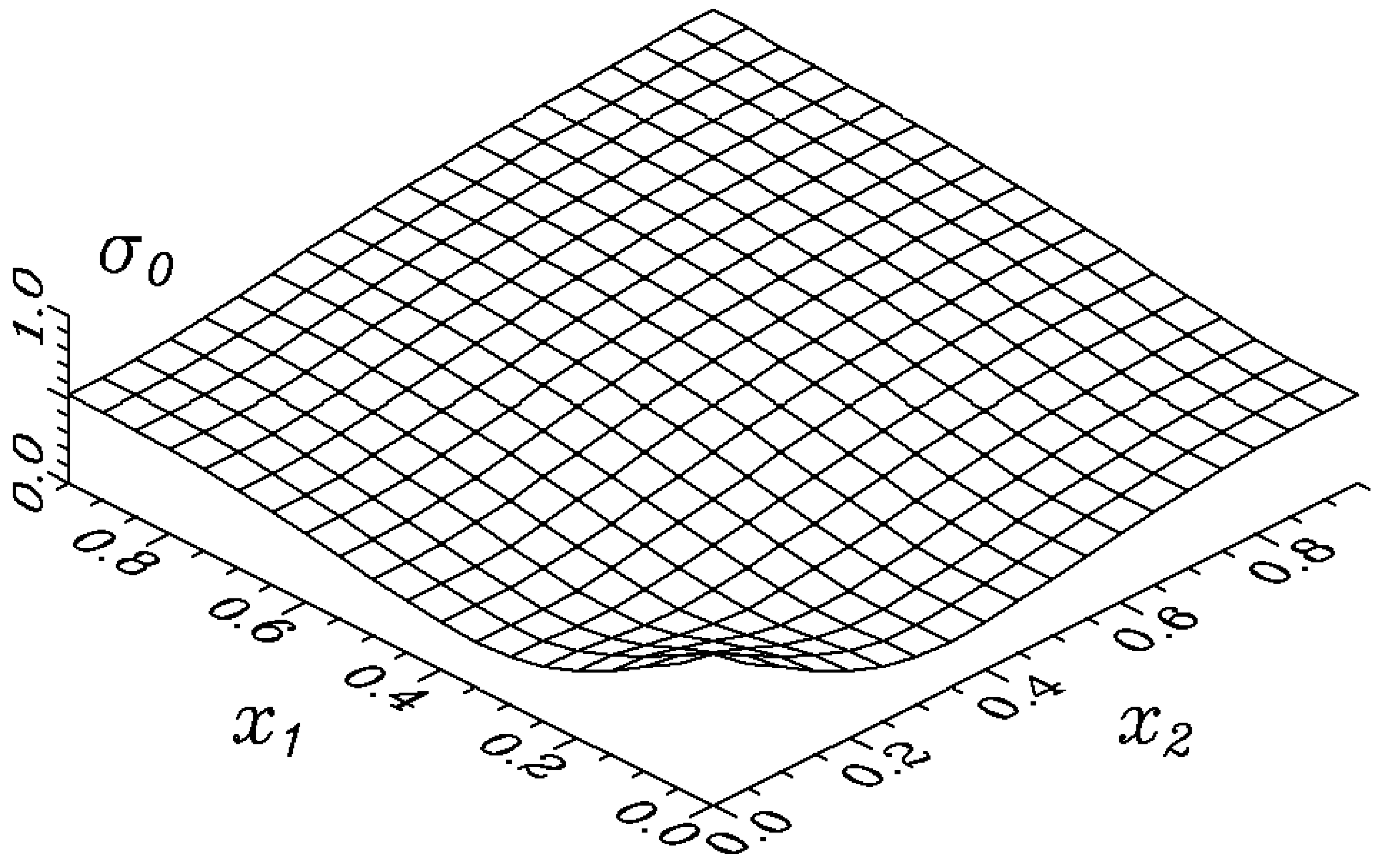

Figure 1. The dependence $\sigma(x_1, x_2)$ of the unbiased r.m.s. deviation of the observations from the AP fit with the borders $x_1 = (t_{*1} - t_1)/(t_n - t_1)$ and $x_2 = (t_{*2} - t_1)/(t_n - t_1)$.

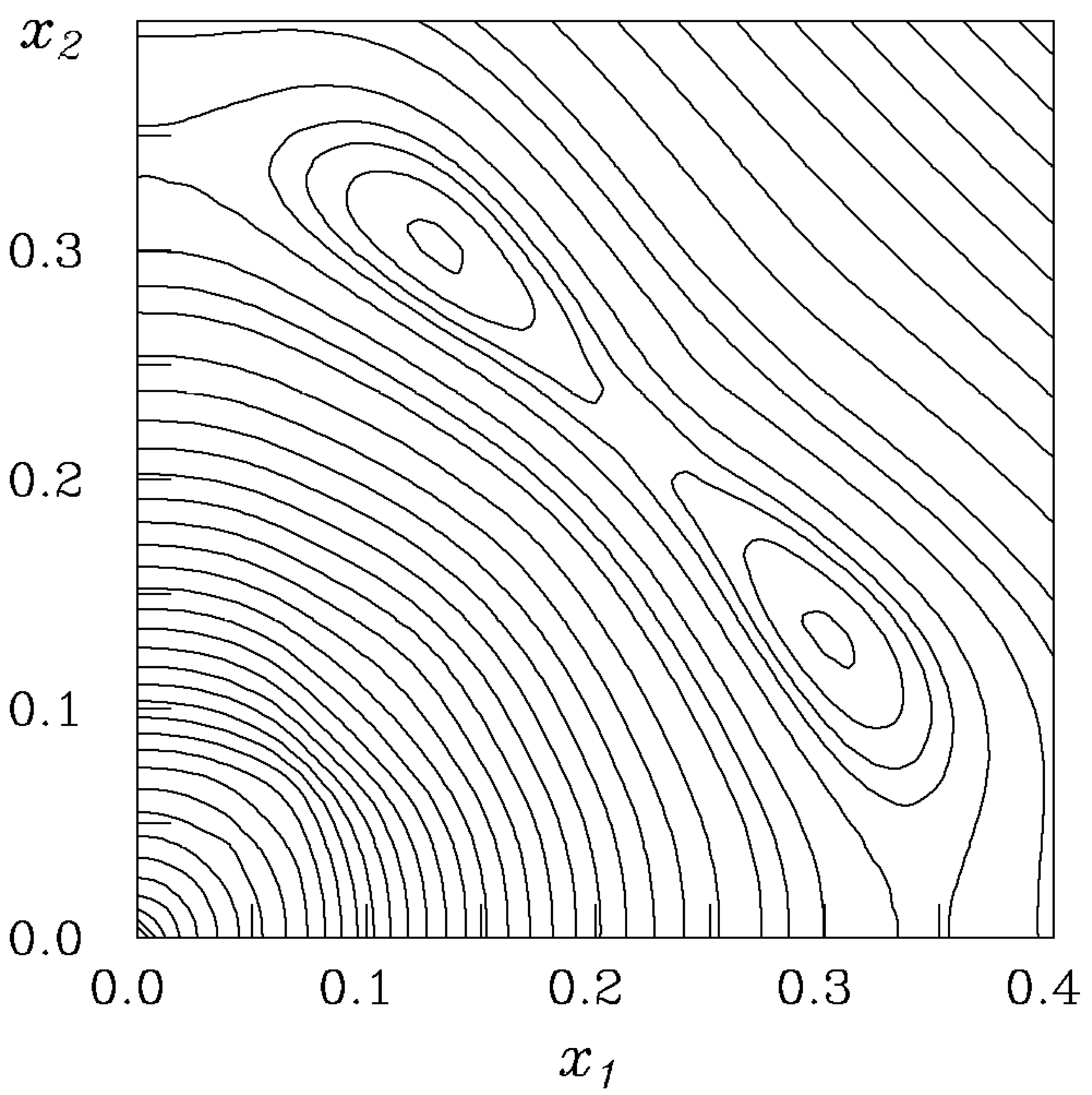

Figure 2. The lines of equal $\sigma(x_1, x_2)$ near the minima of this function.

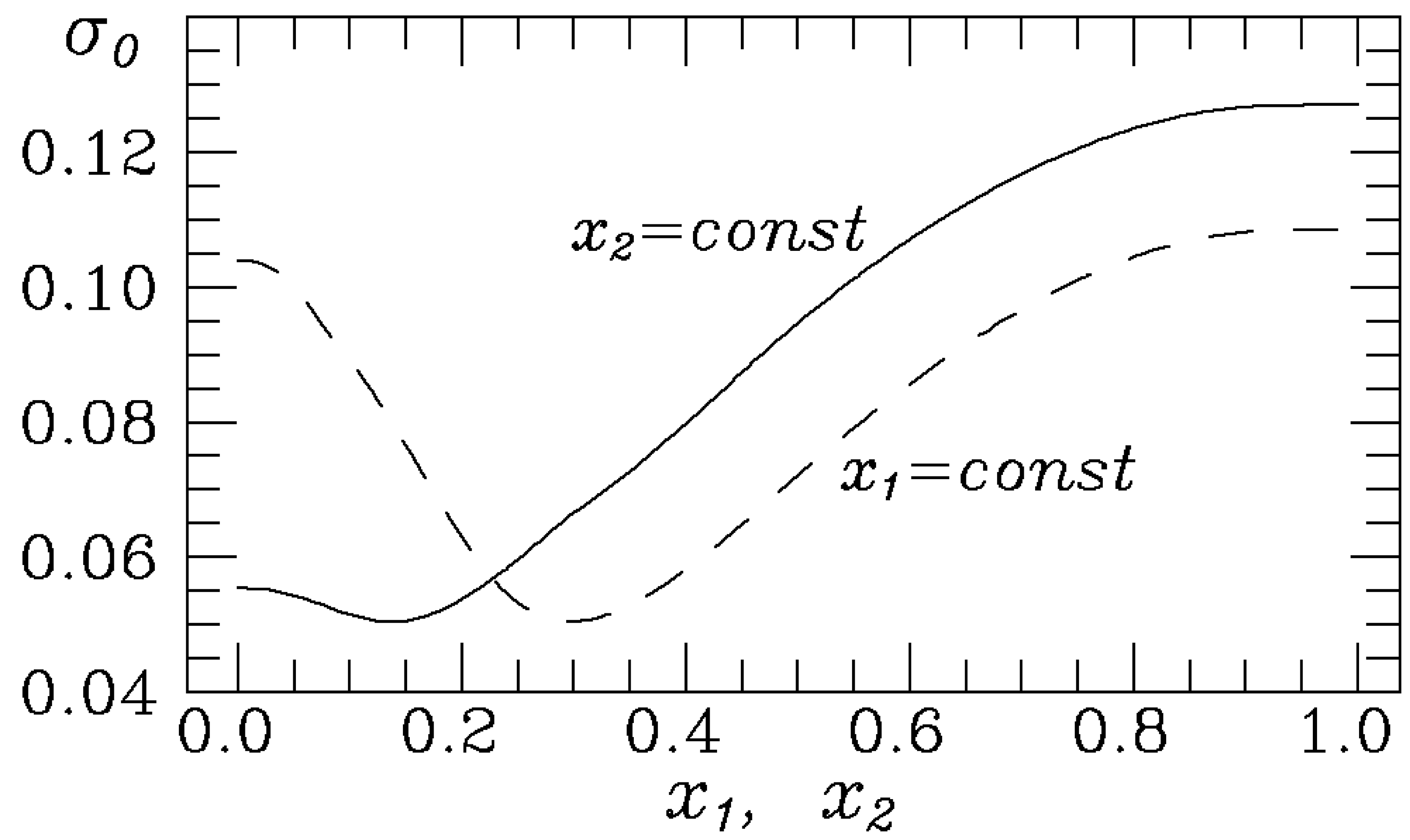

Figure 3. The dependence $\sigma(x_1, x_2)$ of the unbiased r.m.s. deviation of the observations from the AP fit with the borders x_1 and x_2.

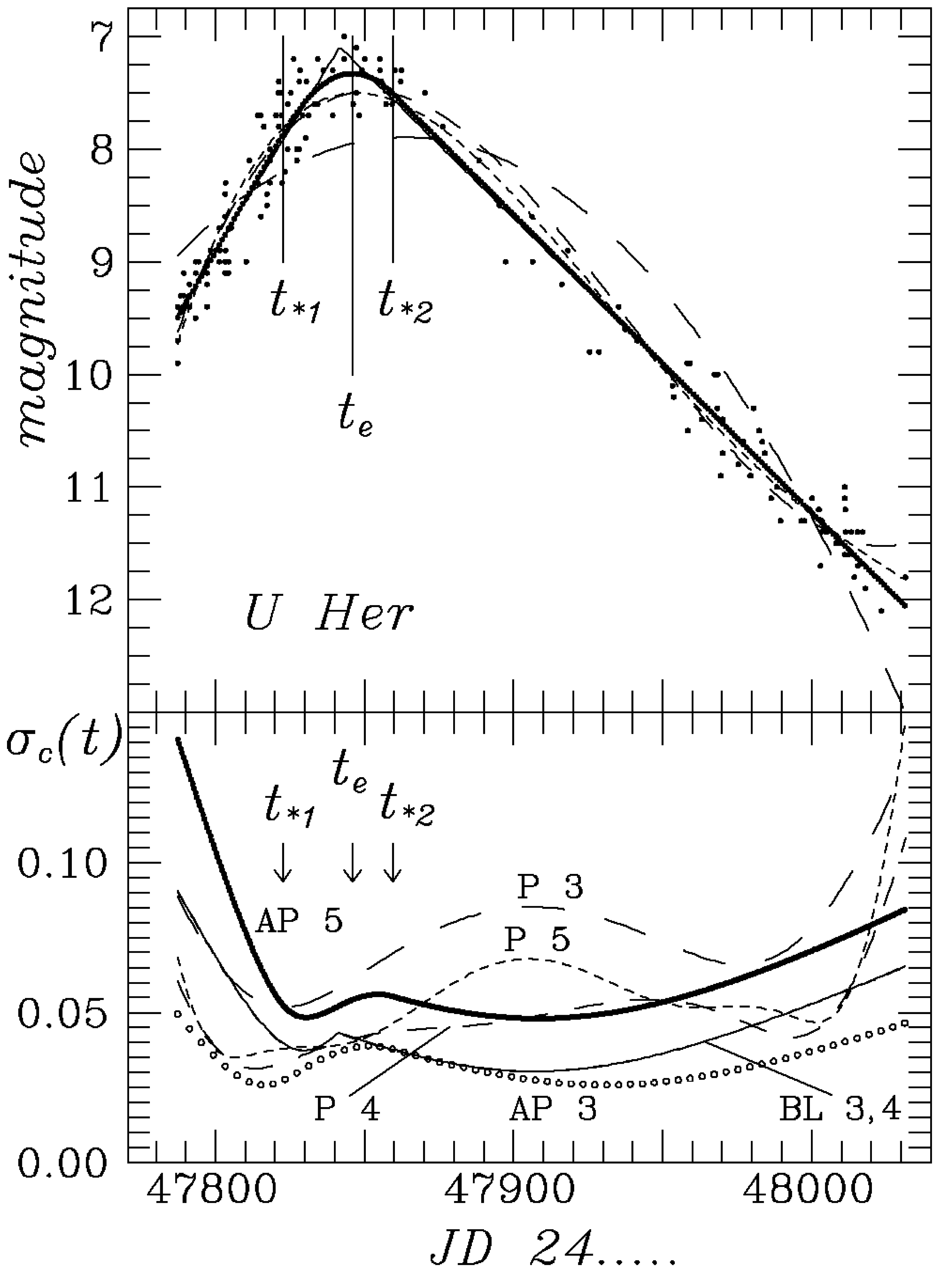

Figure 4. The fits of the data obtained by using different methods (up) and the error estimate of the smoothing function $\sigma_c(t) = \sigma[y_c(t)]$ (bottom).

Table 1. Characteristics of the extremum.

method	t_e	$\sigma[t_e]$	m_e	$\sigma[m_e]$	σ_0
AP 5	47845.68	0.65	7.338	0.054	0.257
AP 3	47845.68	0.21	7.338	0.038	0.257
BL 4	47841.02	1.24	7.095	0.057	0.263
BL 3	47841.02	0.00	7.095	0.043	0.263
P 3	47863.53	2.60	7.895	0.072	0.529
P 4	47851.93	0.83	7.489	0.043	0.291
P 5	47847.22	1.60	7.505	0.041	0.283
RP	47853.12	2.48	7.366	0.045	0.258

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