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T-SOLUTIONS OF THE 5D KALUZA-KLEIN MODEL

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ABSTRACT. We consider spherically-symmetric solution of the 5D Kaluza-Klein theory, which metric coefficients depend on time only. When we construct the appropriate 4+1 splitting of the five-dimensional space and then perform the conformal transformation we get the cosmological model with hypercylinder topology. There are scalar and electromagnetic fields with contact interaction. Besides this, these fields correspond to the inner region of the black hole in the appropriate choice of integration constants. Using 2+2+1 splitting technics and reduction we get the lagrangian of the model. After that we build the canonical formalism of the theory, which admits constraints. These are Hamilton, momentum and Gauss secondary constraints. Momentum constraint is satisfied trivially in the homogeneous case. From the Hamilton constraint we obtain the Einstein-Hamilton-Jacobi equation. 5D metric components arise from the solutions of this equation. Main puprpose of this work is to investigate properties of this metric. It turns out that the configurations with removable and unremovable electric field are possible to exist in this case [Gladush et al., 2015]. Removable electric field can be eliminated with 5D coordinate transformation. Time dependence of the spacetime metric conformal factor is researched. This conformal factor corresponds to the size of the Universe in cosmological model. It turns out that such model describes gravitational collapse of the Universe in the distant future. However, depending on the integration constants signs, there can present some relatively small initial inflation.

Keywords: five-dimensional space-time: Kaluza-Klein model; time-dependent solution: black hole model, cosmological model. **АНОТАЦІЯ.** Будується і досліджується сферично-симетричний розв'язок п'ятивимірної теорії Калуци-Клейна, метричні коефіцієнти якої залежать тільки від часу. При належному 4+1- розщепленні п'ятивимірного простору та конформному відображенні, вони відповідають космологічній моделі з гіперциліндричною топологією зі скалярним та електромагнітним полями, що взаємодіють між собою контактним чином. З

іншого боку, при відповідному виборі сталих інтегрування, вони відповідають внутрішній частині чорної діри п'ятивимірної теорії. За допомогою техніки 2+2+1-розщеплення простору і всіх супутніх величин та відповідної редуції отримується лагранжіан моделі. На основі одержаного лагранжіану будується канонічний формалізм теорії, що містить множники Лагранжа. Цим множникам відповідають вторинні гамільтонова, імпульсна та гаусова в'язі. У однорідному випадку імпульсна в'язь виконується тотожно, а з гамільтонової випливає рівняння Ейнштейна-Гамільтона-Якобі. Із розв'язків цього рівняння отримуємо компоненти метрики п'ятивимірного простору. Основний зміст цієї роботи полягає у вивченні цієї метрики. Виявляється, що у випадку, що розглядається, можливі розв'язки з усуним і не усуним електричним полем [Gladush et al., 2015]. У першому випадку, за допомогою відповідного п'ятивимірного координатного перетворення електричне поле може бути усунуто. Досліджується часова залежність конформного множника просторово-часової метрики, який описує розмір Всесвіту у космологічній моделі. Виявляється, що така модель описує гравітаційний колапс Всесвіту у віддаленому майбутньому. Втім, залежно від знаку сталих інтегрування, у цій космологічній моделі може бути присутнє початкове порівняно незначне розширення.

Ключові слова: п'ятивимірний простір-час: теорія Калуци-Клейна; Т-розв'язок: модель чорної діри, космологічна модель.

Used conventions: Greek indices μ, ν, ρ etc. run from 0 to 3, latin indices from the beginning of alphabet a, b, c etc. get values 0 and 1. Latin indices from the middle of alphabet i, j, k etc. get values 2 and 3.

1. Introduction

In the Kaluza-Klein theory, we start from the 5D space metrics and Einstein-Hilbert action for 5D gravitational field (ψ – 4D scalar field, A_μ – potential of the

electromagnetic field):

$$ds^{(5)2} = g_{\mu\nu} dx^\mu dx^\nu - e^{2\psi} (dz + A_\mu dx^\mu)^2 \quad (1)$$

$$\mathcal{S} = \int \sqrt{-g^{(5)}} \mathcal{R}^{(5)} dz d^4x \quad (2)$$

Here $\mathcal{R}^{(5)}$ and $\sqrt{-g^{(5)}}$ are the scalar curvature and metrics determinant for the 5D manifold, respectively. The last one consists of ordinary 4D space-time and the fifth dimension (which is parametrized by z coordinate). It is also assumed in the Kaluza-Klein theory that there is a Killing vector along this fifth dimension. Because of this assumption, geometrical quantities in our model do not depend on z as soon as such dependence does not have physical manifestation in the theory. From these facts after certain transformations (2) has the following form ($\mathcal{F}_{\mu\nu}$ is an electromagnetic strength tensor):

$$\mathcal{S} = \Delta l \int \left(\tilde{\mathcal{R}}^{(4)} + \frac{3}{2} (\tilde{\nabla}\psi)^2 + \frac{1}{4} e^\psi \mathcal{F}_{\rho\nu} \mathcal{F}^{\rho\nu} \right) \sqrt{-\tilde{g}} d^4x$$

$$\tilde{\mathcal{R}}^{(4)} = \tilde{\mathcal{R}}^{(4)}[\tilde{g}_{\mu\nu}] \quad g_{\mu\nu} = e^{-\psi} \tilde{g}_{\mu\nu} \quad (3)$$

In present article we research the spherically-symmetric solutions of the 5D Kaluza-Klein model, which depends on time only. Such solutions can represent the inner region of the black hole with scalar field or a cosmological model, if the event horizon of that field configuration does not exist.

We are going to work out the field model solutions using Hamilton-Jacobi method and to demonstrate usefulness of that methods for analysis of the homogeneous field configurations. Solution of the same problem can be found in [Hongya Liu et al., 1993], but, contrary to our treatment, that result was obtained through Einstein equations.

2. Initial treatment

In the spherically-symmetric case 4D spacetime metrics in ADM-parametrisation writes:

$$g_{\mu\nu} dx^\mu dx^\nu = e^{-\psi} \left[N^2 dt^2 - e^\lambda (dr + N^r dt)^2 - e^\rho d\sigma^2 \right]$$

$$d\sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2 \quad (4)$$

Here λ , ψ and ρ are the dynamical variables of the model, N and N^r are the Lagrange multipliers. EM field potential has only 2 non-zero components, namely A_1 , which is a dynamical variable, and A_0 – Lagrange multiplier. Therefore, Hamilton function of the model depends on the variables λ , ρ , ψ , A_1 and their conjugate momentums P_λ , P_ρ , P_ψ , P_{EM} . Using (3) and (4), in the spherically-symmetric case we obtain from canonical approach this set of constraints:

$$\mathcal{H}_\perp \approx 0 \quad \mathcal{H}_r \approx 0 \quad \partial_r P_{EM} \approx 0 \quad (5)$$

These equalities manifest that action \mathcal{S} does not depend on time reparametrizations and diffeomorphisms of the r and z coordinates, respectively. Because of the field configuration homogeneity, second and third constraints are satisfied automatically. Particularly, from the third constraint we have that $P_{EM} = Q(t)$, where Q is an electric charge located in the spatial coordinate origin. Therefore, we have only one equation in our model, namely the Hamilton constraint $\mathcal{H}_\perp \approx 0$:

$$\mathcal{H}_\perp = P_\lambda^2 - 2P_\lambda P_\rho + \frac{1}{3} P_\psi^2 + Q^2 e^{\lambda-3\psi} - 4\kappa^2 e^{\lambda+\rho} = 0$$

$$\kappa = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_{r_1}^{r_2} dr \int_{z_1}^{z_2} dz = 4\pi \Delta z \Delta r \quad (6)$$

As we can see, \mathcal{H}_\perp does not contain A_1 , so the corresponding momentum is conserved – $Q = const.$

3. Einstein-Hamilton-Jacobi equation

3.1. Equation

Let us introduce a new set of generalized coordinates in our model:

$$\omega = \lambda - 3\psi \quad \xi = \lambda + \rho \quad \eta = \psi \quad (7)$$

New canonically conjugate momentums:

$$P_\lambda = P_\omega + P_\xi \quad P_\rho = P_\xi \quad P_\psi = -3P_\omega + P_\eta \quad (8)$$

Using (7) and (8) we simplify the expression for Hamilton constraint. Substituting momentums with derivatives of the action \mathcal{S} with respect to the appropriate conjugate coordinates, we get:

$$\left(\frac{\partial \mathcal{S}}{\partial \omega} \right)^2 - \left(\frac{\partial \mathcal{S}}{\partial \xi} \right)^2 + \frac{1}{3} \left(3 \frac{\partial \mathcal{S}}{\partial \omega} - \frac{\partial \mathcal{S}}{\partial \eta} \right)^2 + Q^2 e^\omega - 4\kappa^2 e^\xi = 0 \quad (9)$$

We have obtained the Einstein-Hamilton-Jacobi equation for our field configuration. Here is a solution of this equation:

$$\mathcal{S} = QA_1 + \frac{C_2}{4} \omega + C_2 \eta + s_\xi \int d\xi \sqrt{C_1^2 - 4\kappa^2 e^\xi} +$$

$$+ \frac{1}{2} s_\omega \int d\omega \sqrt{C_1^2 - \frac{1}{12} C_2^2 - Q^2 e^\omega}$$

C_1 and C_2 denote integration constants, s_ω and s_ξ are the sign constants – choosing them be equal to 1 or –1 we can get all 4 families of solutions for (9). As it mentioned in Hamilton-Jacobi method, now we differentiate \mathcal{S} with respect to C_1 , C_2 and Q :

$$\frac{\partial \mathcal{S}}{\partial C_1} = \beta_1 \quad \frac{\partial \mathcal{S}}{\partial C_2} = \beta_2 \quad \frac{\partial \mathcal{S}}{\partial Q} = \beta_Q$$

Here β_1 , β_2 and β_Q are some constants.

$$\begin{aligned}\beta_1 &= s_\xi C_1 \int \frac{d\xi}{\sqrt{C_1^2 - 4\kappa^2 e^\xi}} + \frac{C_1}{2} s_\omega \int F(\omega) d\omega \\ \beta_2 &= \frac{1}{4} \omega + \eta - \frac{1}{24} s_\omega C_2 \int F(\omega) d\omega \\ \beta_Q &= A_1 - \frac{1}{2} s_\omega \int Q F(\omega) e^\omega d\omega\end{aligned}\quad (10)$$

$F(\omega)$ function denotes:

$$F(\omega) = \left(\sqrt{C_1^2 - \frac{1}{12} C_2^2 - Q^2 e^\omega} \right)^{-1}$$

Constants β_1 , β_2 and β_Q describes the initial conditions, that are values of field variables at certain t_0 . As t_0 we can use the event horizon coordinate (see next section) of the field configuration, or just extremum point of the $R(t)$ function.

3.2. Time-dependent solution

Solving (10) with respect to ω , ξ , ρ and A_1 we obtain expressions for three field variables as functions of the fourth one and the integration constants. Particularly, ξ , ρ and A_1 can be expressed as functions of ω . However, it is convenient for further treatment to work out the field potentials as functions of time variable t . Due to the model reparametrization invariance, time coordinate can be chosen arbitrarily. We introduce t variable in the following way:

$$t = \int F(\omega) d\omega \quad (11)$$

We emphasize that time dependency could be chosen differently, that is using momentum definitions in the Lagrange and Hamilton-Jacobi approaches, e. g.:

$$P_\lambda = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = \frac{\partial \mathcal{S}}{\partial \lambda}$$

The left part of this equation contains velocities and multiplier $N(t)$, but the right part consists only of coordinates. Choosing $N(t)$ and solving this differential equation we obtain time dependency.

Using (7), we write down components of the space-time metrics, EM field strength and scalar field as functions of time ($Q \neq 0$):

$$N = \frac{\kappa e^\rho e^{\frac{\lambda-\psi}{2}} \frac{d\rho}{dt}}{\frac{C_2}{4} + \frac{1}{2F(\lambda-3\psi)} s_\omega + s_\xi \sqrt{C_1^2 - 4\kappa^2 e^{\lambda+\rho}}} \quad (12)$$

$$e^{\lambda-\psi} = \frac{(C_1^2 - \frac{1}{12} C_2^2)}{Q^2} \cdot \frac{\exp[2\beta_2 + s_\omega \frac{1}{12} C_2 t]}{\cosh\left(\frac{t}{2} \sqrt{C_1^2 - \frac{1}{12} C_2^2}\right)} \quad (13)$$

$$e^{\rho-\psi} = R^2 = \frac{Q^2 C_1^2}{4\kappa^2 (C_1^2 - \frac{1}{12} C_2^2)} \cdot \frac{\exp[-4\beta_2 - s_\omega \frac{1}{6} C_2 t]}{\cosh^2\left(\beta_1 + s_\xi s_\omega \frac{1}{4} |C_1| t\right)} \quad (14)$$

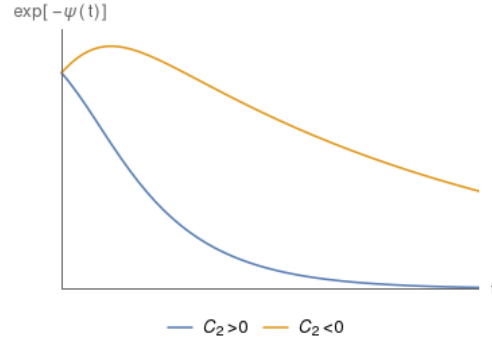


Fig. 1: Qualitative time dependency of the conformal factor $e^{-\psi}$ ($s_\omega = 1$)

$$\mathcal{F}_{01} = \frac{dA_1}{dt} = \frac{C_1^2 - \frac{1}{12} C_2^2}{-2s_\omega Q} \cosh^{-2} \left(\frac{t}{2} \sqrt{C_1^2 - \frac{1}{12} C_2^2} \right) \quad (15)$$

$$\psi = \beta_2 + s_\omega \frac{1}{24} C_2 t + \frac{1}{2} \ln \left[\cosh \left(\frac{t}{2} \sqrt{C_1^2 - \frac{1}{12} C_2^2} \right) \right] \quad (16)$$

Here we chose Lagrange multiplier N^r be equal to zero. As the field configuration is homogeneous such choice does not affect solutions listed above.

4. Solution properties

4.1. Event horizon

By definition, event horizon is an isotropic surface of the scale factor R :

$$(\nabla R)^2 \Big|_{t=t_g} = 0$$

Here $t = t_g$ denotes event horizon coordinate. Besides this, we require that scalar field ψ is equal to zero on the horizon (black hole does not have scalar "hairs"). From the following equalities, we obtain the event horizon coordinate t_g and integration constants condition under which field configuration has such horizon:

$$\frac{1}{N^2} \left(\frac{dR}{dt} \right)^2 \Big|_{t=t_g} = 0 \quad \psi(t = t_g) = 0 \quad (17)$$

From the first equality we have this complex of equations with respect to t_g :

$$\begin{cases} \frac{C_2}{4} + \frac{1}{2} s_\omega \sqrt{C_1^2 - \frac{1}{12} C_2^2 - Q^2 e^{\lambda-\psi} e^{-2\psi}} + \\ + s_\xi \sqrt{C_1^2 - 4\kappa^2 e^{\lambda+\rho}} \Big|_{t=t_g} = 0 \\ \tanh\left(\beta_1 + s_\xi s_\omega \frac{1}{4} |C_1| t_g\right) = -s_\xi \frac{1}{3} \frac{C_2}{|C_1|} \end{cases} \quad (18)$$

The first equation is transcendental, while the second immediately writes:

$$t_g = -\frac{4\beta_1 s_\xi s_\omega}{|C_1|} + s_\xi s_\omega \frac{4}{|C_1|} \operatorname{arcth} \left(-s_\xi \frac{1}{3} \frac{C_2}{|C_1|} \right) \quad (19)$$

Putting (19) to the second equation in (17) (using expression for ψ in (16)) we get the event horizon existence condition for integration constants.

Having (19) and initial data $\psi(t = t_g) = 0$ we get rid of β_1 and β_2 in the expressions for event horizon radius R_g and scalar field:

$$R_g = \frac{QC_1}{2\kappa\sqrt{C_1^2 - \frac{1}{12}C_2^2}} \frac{\cosh\left(\frac{t_g}{2}\sqrt{C_1^2 - \frac{1}{12}C_2^2}\right)}{\cosh\left[\operatorname{arcth}\left(-s_\xi\frac{1}{3}\frac{C_2}{|C_1|}\right)\right]} \quad (20)$$

$$\psi(t) = s_\omega\frac{1}{24}C_2(t - t_g) + \frac{1}{2}\ln\frac{\cosh\left(\frac{t}{2}\sqrt{C_1^2 - \frac{1}{12}C_2^2}\right)}{\cosh\left(\frac{t_g}{2}\sqrt{C_1^2 - \frac{1}{12}C_2^2}\right)} \quad (21)$$

4.2. Singularities of the spacetime geometry

In order to investigate geometrical singularities in present configuration we calculate the Kretschman invariant $K(t)$ for the obtained spacetime metrics (12)-(14). Singularities of this invariant as function of time (let us denote them as t_s) point to the geometrical singularities. By definition:

$$K(t) = \mathcal{R}_{\mu\nu\rho\sigma}^{(4)}\mathcal{R}^{(4)\mu\nu\rho\sigma}$$

It is quite complicated to get it in the direct calculation from spacetime metrics components. Because of this, we apply the 2+2 splitting technique to this issue. Firstly, let us represent spacetime manifold as a tensor product $\mathbb{R}^2 \otimes \mathbb{S}^2$, where the first multiplier (we denote it as $M^{(I)}$) is a two-dimensional manifold with time and radial coordinates, and the second one (which is denoted as $M^{(II)}$) is a two-dimensional sphere with radius $R(t)$. Secondly, we write tensor quantities in the vector basis adapted for splitting. According to this, Kretschman invariant can be written as follows:

$$K(t) = \mathcal{R}_{abde}^{(4)}\mathcal{R}^{(4)abde} + 2\mathcal{R}_{aibj}^{(4)}\mathcal{R}^{(4)aibj} + \mathcal{R}_{ikjl}^{(4)}\mathcal{R}^{(4)ikjl}$$

Using 2+2 splitting, we get:

$$\mathcal{R}_{abde}^{(4)} = \mathcal{R}_{bed}^{(I)f}g_{af} \quad \mathcal{R}_{aibj}^{(4)} = \frac{1}{R}(R_{,a})_{,b}\delta_{ij}$$

$$\mathcal{R}_{ikjl}^{(4)} = -\mathcal{R}_{jlk}^{(II)m}\delta_{im} - \frac{(\nabla R)^2}{R^2}(\delta_{kl}\delta_{ij} - \delta_{jl}\delta_{ik})$$

Here $\mathcal{R}_{bed}^{(I)f}$ and $\mathcal{R}_{jlk}^{(II)m}$ are components of the Riemann tensors of $M^{(I)}$ and $M^{(II)}$, respectively. Covariant derivatives $(R_{,a})_{,b}$ are calculated using metrics on $M^{(I)}$. Finally, we have:

$$K(t) = \frac{4}{R^4}\left[1 + (\nabla R)^2 + (\nabla R)^4\right] + \frac{4}{R^2}(R_{,a})_{,b}(R_{,d})_{,e}g^{ad}g^{be} + \mathcal{R}^{(I)2}(t) \quad (22)$$

Exact expression for the (22) as a function of time is quite long, so we do not show it here. As follows from our analysis, (22) diverges while two conditions are satisfied – when $R(t)$ or $N(t)$ become equal to zero. From (12)-(16) we have that the first condition is fulfilled when $t_s \rightarrow \infty$. It means that point $t \rightarrow \infty$ is a singularity of the black hole. Indeed, scale factor in that point is equal to zero – $R(t \rightarrow \infty) = 0$. So there is a gravitational collapse to the singularity in infinitely distant future for observer in the selected reference frame. In other words, it is geometrically impossible to build a two-dimensional sphere with non-zero radius and volume in such space.

The second condition is fulfilled if t_s satisfies the following equation:

$$-s_\omega\frac{1}{8}C_2 - s_\xi s_\omega\frac{1}{2}|C_1|\tanh\left(\beta_1 + s_\xi s_\omega\frac{1}{4}|C_1|t_s\right) + \frac{1}{4}\sqrt{C_1^2 - \frac{1}{12}C_2^2}\tanh\left(\frac{t_s}{2}\sqrt{C_1^2 - \frac{1}{12}C_2^2}\right) = 0 \quad (23)$$

Conclusion

As it was shown in the analysis above, homogeneous field configuration in the 5D Kaluza-Klein model represents charged black hole with the scalar field inside, or the cosmological model, if event horizon is absent. It follows from the time dependencies of the conformal factor $e^{-\psi(t)}$ (which are shown on fig.1) that such model describes gravitational collapse of a homogeneous universe when $C_2 \geq 0$ and sequential inflation and collapse when $C_2 < 0$.

Solution (12)-(16) contains integration constants C_1 and C_2 , which have the following physical sense. If our field configuration has event horizon, these constants are functions of the central mass, electric and scalar charges of the black hole. Exact view of these functions is a subject for further research.

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