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Functionals, functors and ultrametric spaces

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Abstract We consider different classes of functionals defined on the set of continuous functions on ultrametric spaces. Similarly as in the case of probability measures, idempotent measures, max-min measures and upper semicontinuous capacities we endow the sets of functionals with ultrametrics. We consider some relations between the obtained spaces of functionals. We also discuss the question of completeness.

Keywords Ultrametric space, functional

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1 Introduction

Recall that a metric (resp. a pseudometric) d on a set X is said to be an *ultrametric* (resp. a *pseudoultrametric*) if d satisfies the following strong triangle inequality:

$$d(x,y) \leq \max\{d(x,z), d(z,y)\},\ \ x,y,z \in X.$$

The class of ultrametric spaces is used in the number theory, functional analysis, biology, physics, theoretical computer science. In [6], the ultrametric spaces are applied to the theory of rooted $\mathbb{R}$-trees.

An ultrametric on the set of probability measures of compact support on an ultrametric space is investigated in [11]. This construction was later extended over the cases of idempotent measures, upper semicontinuous capacities and max-min measures. In [8], a fuzzy counterpart of ultrametric is considered on the set of probability measures of compact support.

The present note is devoted to another classes of functionals defined on the sets of continuous functions on ultrametric spaces. We consider some relations between the obtained ultrametric spaces and discuss the question of their completeness.

First, let X be a compact Hausdorff space. As usual, by $C(X)$ we denote the Banach space of continuous real-valued functions on X. The norm in $C(X)$ will be denoted by $\|\cdot\|$.

By $B_r(x)$ we denote the open ball of radius $r > 0$ centered at a point x of a metric space.

We say that a functional $\mu\colon C(X) \to \mathbb{R}$

- *preserves constants* if $\mu(c_X) = c$ for every $c \in \mathbb{R}$;
- *preserves order* if $\mu(\varphi) \leq \mu(\psi)$ whenever $\varphi \leq \psi$;
- *weakly preserves order* if $\mu(a) \leq \mu(\varphi) \leq \mu(b)$ for any function $\varphi \in C(X)$ and constant functions a, b with $a \leq \varphi \leq b$;
- *preserves minima* if $\mu(\min\{\varphi, g\}) = \min\{\mu(\varphi), \mu(g)\}$ for any functions $\varphi, g \in C(X)$;
- *preserves maxima* if $\mu(\max\{\varphi, g\}) = \max\{\mu(f), \mu(g)\}$ for any functions $\varphi, g \in C(X)$;
- weakly preserves minima if $\mu(\min\{\varphi, c\}) = \min\{\mu(\varphi), \mu(c)\}$ for any $\varphi \in C(X)$ and any constant function $c \in C(X)$;
- *weakly preserves maxima* if $\mu(\max\{\varphi, c\}) = \max\{\mu(\varphi), \mu(c)\}$ for any $\varphi \in C(X)$ and any constant function $c \in C(X)$;
- is *additive* if $\mu(\varphi + \psi) = \mu(\varphi) + \mu(\psi)$ for any functions $\varphi, \psi \in C(X)$;
- is *weakly additive* if $\mu(\varphi + c) = \mu(\varphi) + \mu(c)$ for any $\varphi \in C(X)$ and any constant function $c \in C(X)$;
- is *weakly multiplicative* if $\mu(c \cdot \varphi) = \mu(c) \cdot \mu(\varphi)$ for any $\varphi \in C(X)$ and any constant function $c \in C(X)$;
- is *k-Lipschitz* for $k \geq 1$ if $|\mu(\varphi) - \mu(\psi)| \leq k \cdot \|\varphi - \psi\|$ for any functions $\varphi, \psi \in C(X)$.

Now let X be a Tychonov space. We say that a functional $\mu\colon C(X) \to \mathbb{R}$ is of compact support if there is a compact subset A of X with the following property: $\mu(\varphi) = \mu(\psi)$, for any $\varphi, \psi \in C(X)$ with $\varphi|A = \psi|A$. In order to specify this A, one also says that μ is supported on A.

The latter can also be formulated as follows: μ is *supported* on A.

Lemma 1 *If a functional $\mu\colon C(X) \to \mathbb{R}$ is supported on compact subsets $A, B \subset X$, then also μ is supported on $A \cap B$.*

Proof Suppose that $\varphi, \psi \in C(X)$ are such that $\varphi|(A \cap B) = \psi|(A \cap B)$.

Proof By the Tietze Extension Theorem, there exists a function $\varphi_1 \in C(X)$ such that $\varphi_1|A = \varphi|A$ and $\varphi_1|B = \psi|B$. Then $\mu(\varphi) = \mu(\varphi_1) = \mu(\psi)$.

The same definitions but the last one can be extended over the functionals defined on $C(X)$, for any Tychonov space X. The only minor difficulty arises when we consider the k-Lipschitz functionals, as the space $C(X)$ is not normed for noncompact X.

Let us consider the spaces of functionals with compact support on $C(X)$. In this case, there is no problem in defining k-Lipschitz functionals as one can consider the restrictions of the functions onto a suitable compact subset of X and to use the sup-norm of the restrictions in the definition.

2 Ultrametrization

Now, let (X, d) be an ultrametric space. Following [4] we denote, for any $r > 0$, by $\mathcal{F}_r = \mathcal{F}_r(X)$ the set of functions on X that are constant on all balls of radius r. Given a set of functionals $F(X)$ on $C(X)$, we let, for every $\mu, \nu \in F(X)$,

$$\hat{d}(\mu, \nu) = \inf\{r > 0 \mid \mu(\varphi) = \nu(\varphi), \text{ for every } \varphi \in \mathcal{F}_r\}$$

(convention: $\inf \emptyset = +\infty$).

Proposition 1 *Suppose that $\hat{d}(\mu, \nu) < +\infty$, for every $\mu, \nu \in F(X)$. Then $\hat{d}$ is a pseudoultrametric on $F(X)$.*

Proof It suffices to prove the strong triangle inequality. Let $\mu, \nu, \tau \in F(X)$. If $\hat{d}(\mu, \tau) \leq r$, $\hat{d}(\nu, \tau) \leq r$, then, for any $s > r$ and every $\varphi \in \mathcal{F}_s$, we see that $\mu(\varphi) = \tau(\varphi) = \nu(\varphi)$, whence $\hat{d}(\mu, \nu) \leq s$. Therefore, $\hat{d}(\mu, \nu) \leq r$.

Proposition 2 *Suppose that every $\mu \in F(X)$ preserves order and is weakly additive. Then $\hat{d}$ is an ultrametric on the set $F(X)$.*

Proof Suppose the contrary. Then there exist distinct $\mu, \nu \in F(X)$ such that $\hat{d}(\mu, \nu) = 0$. We may suppose that the supports of μ, ν are contained in a compact space $Y \subset X$. Without loss of generality, one may suppose that $X = Y$.

There exists $\varphi \in C(X)$ such that $\mu(\varphi) \neq \nu(\varphi)$. We may suppose that $\mu(\varphi) + c \leq \nu(\varphi)$, where $c > 0$. Because of compactness of Y, there exists $r > 0$ and a function $\psi \in \mathcal{F}_r$ such that $\psi \leq \varphi$ and $\|\psi - \varphi\| < c/2$. Then, by the weak additivity,

$$\mu(\psi) \leq \mu(\varphi) \leq \mu(\psi + (c/2)) = \mu(\psi) + (c/2),$$
$$\nu(\psi) \leq \nu(\varphi) \leq \nu(\psi + (c/2)) = \nu(\psi) + (c/2)$$

and, since $\mu(\psi) = \nu(\psi)$, $\mu(\psi + c/2) = \nu(\psi + c/2)$, we conclude that $|\mu(\psi) - \mu(\varphi)| < c/2$ therefore obtaining a contradiction.

Let $O(X)$ denote the set of order-preserving weakly additive functionals [9] of compact support on $C(X)$ and let $S(C)$ denote the subset of $O(X)$ consisting of weakly multiplicative functionals [10].

Proposition 3 *For any ultrametric space (X, d), the set $S(X)$ is closed in the space $O(X)$.*

Proof Suppose that $\mu \in O(X) \setminus S(X)$. Then there exist $\varphi \in C(X)$ and $c \in \mathbb{R}$ such that $\mu(c\varphi) \neq c\mu(\varphi)$. Without loss of generality, one may assume that $\varphi \geq 0$. Indeed, assuming, as above, that X is compact, one can find $a \in \mathbb{R}$ such that $\varphi + a \geq 0$. Then

$$\mu(c(\varphi + a)) = \mu(c\varphi) + ca \neq c\mu(\varphi) + ca = c(\mu(\varphi) + a) = c\mu(\varphi + a).$$

First, suppose that $\mu(c\varphi) < c\mu(\varphi)$. Then $\mu(c\varphi) < (c/\alpha)\mu(\varphi)$, for some $\alpha > 1$. There exists $\psi \in \mathcal{F}_r$, for some $r > 0$, such that $\psi \leq \varphi \leq \alpha\psi$.

Then

$$\mu\left(\frac{c}{\alpha}(\alpha\psi)\right)\mu(c\psi) \leq \mu(c\varphi) < \frac{c}{\alpha}\mu(\varphi) \leq \frac{c}{\alpha}\mu(\alpha\psi).$$

Since $\alpha\psi \in \mathcal{F}_r$, we conclude that, for every $\nu \in O(X)$ with $\hat{d}(\mu, \nu) < r$,

$$\nu\left(\frac{c}{\alpha}(\alpha\psi)\right) = \mu\left(\frac{c}{\alpha}(\alpha\psi)\right)\mu(c\psi) < \frac{c}{\alpha}\mu(\alpha\psi) = \frac{c}{\alpha}\nu(\alpha\psi),$$

and therefore $\nu \in O(X) \setminus S(X)$.

Similar arguments can be applied to the case when $\mu(c\varphi) > c\mu(\varphi)$. Thus, the set $O(X) \setminus S(X)$ is open in $O(X)$.

It is well-known that the functor of order-preserving functionals in the category **Comp** of compact Hausdorff spaces preserves intersections [9]. This allows us to define the notion of support $\mathrm{supp}(\mu)$ of any $\mu \in O(X)$ in a standard way: $\mathrm{supp}(\mu)$ is the minimal closed subset A in X such that $\mu(\varphi) = \nu(\varphi)$ whenever μ and ν agree on A.

As usual, we denote by $\exp X$ the set of nonempty compact subsets in a topological space X. If (X, d) is a metric space, we endow $\exp X$ with the Hausdorff metric d_H:

$$d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}.$$

Recall (see, e.g., [7] for details) that a function c defined on the closed subsets of X and taking its values in $[0, 1]$ is called an *upper semicontinuous capacity* on a space X if the following conditions are satisfied:

1. $c(\emptyset) = 0$ and $c(X) = 1$;
2. there exists $A \in \exp X$ such that $c(B) = c(B \cap A)$, for every closed set $B \subset X$;
3. $c(B) \leq c(C)$, whenever B, C are closed subsets in X and $B \subset C$;
4. for every $a > 0$ and every compact subset B in X with $c(B) < a$ there exists a neighborhood U of B such that, for every closed subset C in X with $C \subset U$, we have $c(C) < a$.

Every capacity c determines the so-called *Choquet integral* as follows:

$$\int_X \varphi dc = \int_0^\infty c(\varphi \geq t)dt + \int_{-\infty}^0 (c(\varphi \geq t) - 1)dt,$$

where $\varphi \in C(X)$.

The functional $\mu_c \colon \varphi \mapsto \int_X \varphi dc$ is known to satisfy the following properties [7]:

1. μ_c is comonotonically additive, i.e., $\mu_c(\varphi + \psi) = \mu_c(\varphi) + \mu_c(\psi)$, for every comonotone functions $\varphi, \psi \in C(X)$ (i.e., functions $\varphi, \psi \in C(X)$ such that $\varphi(x) - \varphi(y))(\psi(x) - \psi(y)) \geq 0$, for every $x, y \in X$);
2. μ_c is order-preserving.

Note that the comonotonical additivity implies the weak additivity, since any constant function is comonotone with every another continuous function. In addition, μ_c preserves constants. We conclude that $M(X) \subset O(X)$.

Proposition 4 *The map* $\mathrm{supp} \colon O(X) \to \exp X$ *is not, in general, nonexpanding.*

Proof This follows from the fact that the map $\mathrm{supp} \colon M(X) \to \exp X$ is not, in general, nonexpanding (see [4]).

Proposition 5 *The set* $M(X)$ *is closed in the space* $O(X)$.

Proof Suppose that $\mu \in O(X) \setminus M(X)$. Then μ is not comonotonically additive, i.e., there exist comonotone functions $\varphi, \psi \in C(X)$ such that $\mu(\varphi + \psi) \neq \mu(\varphi) + \mu(\psi)$.

Since we are dealing with functionals of compact support, we may assume, without loss of generality, that the space X is compact. Also, we assume that $\mu(\varphi) + \mu(\psi) - \mu(\varphi + \psi) = \varepsilon > 0$. The case of the reverse inequality can be treated similarly.

There exists $r > 0$ and a finite set $\{x_1, \ldots, x_n\} \subset X$ such that $\{B_r(x_i) \mid i = 1, \ldots, n\}$ is a disjoint cover of X and the oscillation of φ and ψ on every set $B(x_i)$ is less then $\varepsilon/5$.

Define functions $\varphi', \psi' \colon X \to \mathbb{R}$ as follows: $\varphi'(x) = \varphi(x_i)$, $\psi'(x) = \psi(x_i)$, whenever $x \in B(x_i)$. First remark that φ' and ψ' are comonotone. Indeed, given $x, y \in X$, find i, j such that $x \in B_r(x_i)$, $y \in B_r(x_j)$; then

$$(\varphi'(x) - \varphi'(y))(\psi'(x) - \psi'(y)) = (\varphi(x_i) - \varphi'(x_j))(\psi(x_i) - \psi(x_j)) \geq 0.$$

We have $\varphi' \geq \varphi - (\varepsilon/5)$, $\psi' \geq \psi - (\varepsilon/5)$, $\varphi' + \psi' \geq \varphi + \psi - (2\varepsilon/5)$. Then

$$\mu(\varphi')+\mu(\psi')-\mu(\varphi'+\psi') \geq \mu(\varphi)+\mu(\psi)-(2\varepsilon/5)-\mu(\varphi'+\psi')-(2\varepsilon/5) = \varepsilon-(4\varepsilon/5) > 0$$

and therefore $\mu(\varphi') + \mu(\psi') \neq \mu(\varphi' + \psi')$.

Note that $\varphi', \psi', \varphi' + \psi' \in \mathcal{F}_r$ and therefore, for any $\nu \in O(X)$ with $\hat{d}(\mu, \nu) < r$, we have $\nu \in O(X) \setminus M(X)$.

Applying results of the paper [4] we conclude that, in general, the space $O(X)$ is not necessarily complete even for complete X.

Denote by **UMetr** the category of ultrametric spaces and nonexpanding maps. Let $f \colon X \to Y$ be a morphism in **UMetr**. Define a map $O(f) \colon O(X) \to O(Y)$ as follows: $O(f)(\mu)(\varphi) = \mu(\varphi f)$, for every $\mu \in O(X)$ and $\varphi \in C(Y)$.

Proposition 6 *The map $O(f)$ is nonexpanding.*

Proof We denote by d and ϱ the ultrametrics on X and Y respectively. Note that, given $r > 0$, we have $\varphi f \in \mathcal{F}_r(X)$, for every $\varphi \in \mathcal{F}_r(Y)$. Then, for any $\mu, \nu \in O(X)$ with $\hat{d}(\mu, \nu) < r$ and any $\varphi \in \mathcal{F}_r(Y)$ we obtain

$$O(f)(\mu)(\varphi) = \mu(\varphi f) = \nu(\varphi f) = O(f)(\nu),$$

and therefore $\hat{\varrho}(O(f)(\mu), O(f)(\nu)) < r$.

We therefore obtain a functor in the category **UMetr**; we keep the notation O for this functor. One can define similarly a subfunctor S of O.

Denote by $O_\omega(X)$ the set of functionals of finite support in $O(X)$.

Proposition 7 *The set $O_\omega(X)$ is dense in $O(X)$.*

Proof Let $\mu \in O(X)$ and $r > 0$. Since the set $\operatorname{supp}(\mu)$ is compact, one may assume that $X = \operatorname{supp}(\mu)$. There exists a finite set Y in X such that the family $\{B_r(y) \mid y \in Y\}$ is a disjoint cover of X. Denote by $f \colon X \to Y$ the retraction that sends every ball to its center. Let $\nu = O(f)(\mu)$. Then $\nu \in O_\omega(X)$ and it is clear that $\hat{d}(\mu, \nu) \leq r$.

One can prove a similar statement for the set $S(X)$ as well as for another sets of functionals.

3 Remarks and open problems

Investigate the pseudoultrametrics on the sets of functionals defined at the beginning of this note.

Investigate the constructions S and O in the realm of fuzzy ultrametric spaces.

It is reasonable to consider the following pseudoultrametric on the sets of functionals:

$$\tilde{d}(\mu,\nu) = \max\{\hat{d}(\mu,\nu), d_H(\mathrm{supp}(\mu),\mathrm{supp}(\nu))\}.$$

We conjecture that in the case of functor O and some of their subfunctors (e.g., M and S) the obtained metrization preserves completeness.

We did not consider here algebraic properties of the functors generated by functionals, in particular, the functors S and O. The previous publications show that these properties can substantially differ. In particular, unlikely to the case of functors of probability measures, idempotent measures and max-min measures, the capacity functor does not generate a monad in the category **UMetr**.

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