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On conformally reducible pseudo-Riemannian spaces

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Abstract. The present paper studies the main type of conformal reducible conformally flat spaces. We prove that these spaces are subprojective spaces of Kagan, while Riemann tensor is defined by a vector defining the conformal mapping. This allows to carry out the complete classification of these spaces. The obtained results can be effectively applied in further research in mechanics, geometry, and general theory of relativity. Under certain conditions the obtained equations describe the state of an ideal fluid and represent quasi-Einstein spaces. Research is carried out locally in tensor shape.

Анотація. В роботі розглянуто основний тип конформно-звідних конформно пласких просторів. Доведено, що такі простори будуть субпроективними просторами Кагана, при чому тензор Рімана визначається вектором, що задає конформне відображення. Це дозволило навести повну класифікацію таких просторів. Результати роботи можуть ефективно застосовуватись для подальших досліджень в механіці, геометрії та загальній теорії відносності. Отримані рівняння при певних умовах є рівняннями стану ідеальної рідини та характеризують квазі-Ейнштейнові простори. Дослідження ведуться локально, в тензорній формі.

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1. Introduction

Riemannian or pseudo-Riemannian spaces (shortly Riemannian spaces) which are related by a conformal mappings have been studied by many geometers, see e.g.[10–12].

For the sake of examining such pairs of spaces, H. Weyl [22] introduced a special tensor being invariant under conformal mappings, which was called the *tensor of conformal curvature.* Weyl and Schouten [19] studied problems concerning conformal mappings of Riemannian spaces onto flat spaces, and they obtained necessary and sufficient conditions of tensor character for a Riemannian space to be conformally flat, see [7].

H. Brinkmann [1] and others, e.g. [19], investigated existence of conformal mappings of Riemannian spaces onto Einstein spaces. Particularly, H. Brinkmann [1] proved that any conformal mappings between Einstein spaces is concircular, see[3, 9].

As far as other special types of conformal mappings are concerned, let us mention the following. V. F. Kagan [8] studied spaces admitting such a mapping onto Euclidean space under which geodesics are mapped onto curves that are involved in two-dimensional subspaces passing through a general point. The case when this point is an ideal point (point "at infinity"), i.e. the situation when all two-dimensional subspaces are parallel to some direction, was studied by G. Vranceanu [21]. The spaces with this property are called *subprojective spaces of Kagan*, and the mappings mentioned above is a composition of a conformal and a geodesic mapping.

Under a *geodesic circle* we will mean a curve for which the first curvature is constant and the second curvature is zero. A conformal mappings preserving geodesic circles is called *concircular*. When K. Yano [23–26] investigated such mappings he has introduced the concept of the tensor of concircular curvature, invariant under concircular mappings. It turns out that invariance of the tensor of concircular curvature under a conformal mapping is a necessary and sufficient condition for the mapping to be concircular, see [19].

In this respect, we can also mention the works of S. Fedishenko [5, 6] who studied conformal mappings preserving the Ricci tensor and the scalar curvature, respectively.

The foregoing discussion made it clear that every pair of pseudo-Riemannian spaces can be connected by conformal mapping. The aim of our article is a research on spaces permitting conformal mappings on almost reducible spaces.

2. Conformal mappings of Riemannian manifolds

Let V_n, $(n > 2)$, be a pseudo-Riemannian space with metric tensor $g_{ij}(x)$ and $\bar{V}_n$ be a pseudo-Riemannian space with metric tensor $\bar{g}_{ij}(x)$. A conformal mapping is a diffeomorphism of V_n onto $\bar{V}_n$ such that

$$\bar{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x), \tag{2.1}$$

where σ is a function on V_n. If σ is constant then it is a homothetic mapping.

From (2.1) we obtain

$$\bar{g}^{ij} = e^{-2\sigma} g^{ij},$$

where g^{ij} and $\bar{g}_{ij}$ are inverse matrices of the metric tensors on V_n and $\bar{V}_n$, respectively.

We have the following formulas for Christoffel symbols:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}, \tag{2.2}$$

for Riemann tensor:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + g^{h\alpha}(\sigma_{\alpha h} g_{ij} - \\ - \sigma_{\alpha j} g_{ik}) + \Delta_1 \sigma (\delta^h_k g_{ij} - \delta^h_j g_{ik}),$$

for Ricci tensor

$$\bar{R}_{ij} = R_{ij} + (n-2)\sigma_{ij} + (\Delta_2 \sigma + (n-2)\Delta_1 \sigma) g_{ij},$$

and for scalar curvature

$$\bar{R} = e^{-2\sigma}(R + 2(n-1)\Delta_2 \sigma + (n-1)(n-2)\Delta_1 \sigma).$$

Here and below Γ^h_{ij} denote the Christoffel symbols of second type, R^h_{ijk} are components of the Riemannian tensor, R_{ij} are components of the Ricci tensor defined by $R_{ij} = R^\alpha_{ij\alpha}$, $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature, δ^i_j is the Kronecker symbol, $\sigma_i \equiv \frac{\partial \sigma}{\partial x^i} \equiv \sigma_{,i}$, $\sigma^h = \sigma_\alpha g^{\alpha h}$,

$$\sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j}, \tag{2.3}$$

$\Delta_1 \sigma$ and $\Delta_2 \sigma$ are the first and second Beltrami's symbols which are determined by $\Delta_1 \sigma = g^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}$, $\Delta_2 \sigma = g^{\alpha\beta}\sigma_{,\alpha\beta}$, and *comma* is the sign of the covariant derivative according to the Riemannian connection in V_n.

Objects of conformally related spaces V_n and $\bar{V}_n$ will be distinguished by bar.

3. Conformally almost reducible spaces

A space V_n is called a *conformally reducible*, whenever its metric in a certain holonomic system of coordinates takes the following form, [2,4,18]:

$$ds^2 = e^{-2\sigma} \sum_{k=1}^{r} ds_k^2,$$

here ds_k^2 is defined as a quadratic form of a space

$$V_{m_k},\ (m_1 + m_2 + \ldots + m_r = n),\ r > 1,$$

and $\sigma = \sigma(x^1, x^2, \ldots, x^n)$ is a certain invariant. In order to find a tensor characteristic of conformally reducible spaces, let us turn our attention to two possible cases.

The first case is when $r = 2$, i.e.

$$ds^2 = e^{-2\sigma}(ds_1^2 + ds_2^2),$$

where

$$\begin{aligned} ds_1^2 &= g_{i_1 j_1}(x^1, x^2, \ldots, x^p) dx^{i_1} dx^{j_1}, \qquad && (i_1, j_1 = 1, 2, \ldots, p), \\ ds_2^2 &= g_{i_2 j_2}(x^{p+1}, x^{p+2}, \ldots, x^n) dx^{i_2} dx^{j_2}, && (i_2, j_2 = p+1, p+2, \ldots, n). \end{aligned}$$

This case is called the *main* type.

In order to define a pseudo-Riemannian space as conformally reducible of main type, it is necessary and sufficient, that it contains a symmetric (non proportional to metric) tensor c_{ij}, which jointly with some invariant σ complies with the following conditions:

$$c_{i\alpha} c_j^\alpha = c_{ij}, \qquad c_{ij,k} = -(\sigma_i c_{jk} + \sigma_j c_{ik}) + \sigma_\alpha (c_i^\alpha g_{jk} + c_j^\alpha g_{ik}), \tag{3.1}$$

where $\sigma_i = \partial_i \sigma$, $c_j^i = g^{\alpha i} c_{\alpha j}$.

Integrability conditions of equations (3.1) taking into account Ricci's identity, take the following form:

$$\begin{aligned} c_{i\alpha} R^\alpha_{jkl} + c_{j\alpha} R^\alpha_{ikl} = \sigma_{jk} c_{li} - \sigma_{jl} c_{ki} + \sigma_{ik} c_{lj} - \sigma_{il} c_{kj} + \\ + \sigma_{\alpha l}(g_{ik} c_j^\alpha + g_{jk} c_i^\alpha) - \sigma_{\alpha k}(g_{il} c_j^\alpha + g_{jl} c_i^\alpha) + \\ + \Delta_1 \sigma (g_{jl} c_{ik} - g_{jk} c_{il} + g_{il} c_{jk} - g_{ik} c_{jl}). \end{aligned} \tag{3.2}$$

Cycling (3.2) by indices $(i\,k\,l)$ we get:

$$\begin{aligned} & c_{i\alpha} R^\alpha_{jkl} + c_{k\alpha} R^\alpha_{jli} + c_{l\alpha} R^\alpha_{jik} = \\ & \quad = (\sigma_{\alpha l} c_i^\alpha - \sigma_{\alpha i} c_l^\alpha)\, g_{jk} - (\sigma_{\alpha k} c_i^\alpha - \sigma_{i\alpha} c_k^\alpha)\, g_{jl} - (\sigma_{\alpha l} c_k^\alpha - \sigma_{k\alpha} c_l^\alpha)\, g_{ij}. \end{aligned}$$

Multiplying the latter identity by g^{kj} and wrapping it by indices k, j we will obtain that

$$c_{i\alpha} R^\alpha_l - c_{l\alpha} R^\alpha_i = (n-2)\left(\sigma_{\alpha l} c_i^\alpha - \sigma_{\alpha i} c_l^\alpha\right), \tag{3.3}$$

where $R^i_l = R_{\alpha l} g^{\alpha i}$.

Thus, the following statement holds:

Lemma 3.1. *In conformal reducible spaces the condition* (3.3) *holds for tensors* c_{ij} *and* σ_{ij}.

If a pseudo-Riemannian space V_n, $(n > 2)$, admits conformal mappings to the flat space, then it is called *conformally flat*.

Such space is characterized by the following conditions:

$$R_{hijk} = P_{hk} g_{ij} - P_{hj} g_{ik} + P_{ij} g_{hk} - P_{ik} g_{hj}, \tag{3.4}$$

$$P_{ij,k} - P_{ik,j} = 0, \tag{3.5}$$

where

$$P_{ij} = \tfrac{1}{n-2}\left(R_{ij} - \tfrac{1}{2(n-1)} R g_{ij}\right). \tag{3.6}$$

Conditions (3.4) and (3.5) are necessary and sufficient for a pseudo-Riemannian space V_n to be conformally flat, [15, 16].

Let us note that (3.4) is true for every three-dimensional pseudo-Riemannian space.

Moreover, conformally flat spaces constitute in fact subclass of a class of conformally reducible pseudo-Riemannian spaces.

Consider now a tensor characteristic for conformal reducibility of conformally flat spaces. In order to get it let us substitute (3.4) into (3.2). Then we get

$$\begin{gathered} \underset{1}{\Omega}_{il} g_{jk} - \underset{1}{\Omega}_{ik} g_{jl} + \underset{1}{\Omega}_{jl} g_{ik} - \underset{1}{\Omega}_{jk} g_{il} + \\ \underset{2}{\Omega}_{il} c_{jk} - \underset{2}{\Omega}_{ik} c_{jl} + \underset{2}{\Omega}_{jl} c_{ik} - \underset{2}{\Omega}_{jk} c_{il} = 0, \end{gathered} \tag{3.7}$$

where

$$\underset{1}{\Omega}_{ij} = c_{\alpha i} P^{\alpha}_{j} - \sigma_{\alpha j} c^{\alpha}_{i} + \Delta_1 \sigma c_{ij}, \tag{3.8}$$

$$\underset{2}{\Omega}_{ij} = -P_{ij} + \sigma_{ij}. \tag{3.9}$$

Using (3.6), we get from (3.3) that

$$c_{i\alpha} P^{\alpha}_{l} - c_{l\alpha} P^{\alpha}_{i} = \sigma_{\alpha l} c^{\alpha}_{i} - \sigma_{\alpha i} c^{\alpha}_{l}.$$

Hence it follows from (3.8) that the tensor $\underset{1}{\Omega}_{ij}$ is symmetrical, namely

$$\underset{1}{\Omega}_{ij} = \underset{1}{\Omega}_{ji}.$$

This property is also true for the tensor $\underset{2}{\Omega}_{ij}$.

Theorem 3.2. *Conformally flat conformally reducible pseudo-Riemannian spaces comply to the following condition:*

$$R_{hijk} = \sigma_{hk}g_{ij} - \sigma_{hj}g_{ik} + \sigma_{ij}g_{hk} - \sigma_{ik}g_{hj} - \frac{2}{n}\underset{2}{\Omega}\,(g_{hk}g_{ij} - g_{hj}g_{ik}), \quad (3.10)$$

where $\underset{2}{\Omega} = \underset{2}{\Omega}_{\alpha\beta}g^{\alpha\beta}$.

Proof. Alternating (3.7) by indices $(j\,l)$ we get:

$$\underset{1}{\Omega}_{il}g_{jk} - \underset{1}{\Omega}_{jk}g_{il} + \underset{2}{\Omega}_{il}c_{jk} - \underset{2}{\Omega}_{jk}c_{il} - \underset{1}{\Omega}_{ij}g_{lk} + \underset{1}{\Omega}_{lk}g_{ij} - \underset{2}{\Omega}_{ij}c_{lk} + \underset{2}{\Omega}_{lk}c_{ij} = 0.$$

Renaming in the previous identity the indices i and l we also obtain that

$$\underset{1}{\Omega}_{li}g_{jk} - \underset{1}{\Omega}_{jk}g_{li} + \underset{2}{\Omega}_{li}c_{jk} - \underset{2}{\Omega}_{jk}c_{li} - \underset{1}{\Omega}_{lj}g_{ik} + \underset{1}{\Omega}_{ik}g_{lj} - \underset{2}{\Omega}_{lj}c_{ik} + \underset{2}{\Omega}_{ik}c_{lj} = 0.$$

Adding this identity to (3.7) gives

$$\underset{1}{\Omega}_{il}g_{jk} - \underset{1}{\Omega}_{jk}g_{il} + \underset{2}{\Omega}_{il}c_{jk} - \underset{2}{\Omega}_{jk}c_{il} = 0. \quad (3.11)$$

Now let us wrap (3.11):

$$\underset{1}{\Omega}_{il} = \frac{\underset{1}{\Omega}}{n}g_{il} + \frac{\underset{2}{\Omega}}{n}c_{il} - \frac{c}{n}\underset{2}{\Omega}_{il},$$

where

$$\underset{1}{\Omega} = \underset{1}{\Omega}_{\alpha\beta}g^{\alpha\beta}, \qquad c = c_{\alpha\beta}g^{\alpha\beta}.$$

Then (3.11) transforms into

$$c_{il}\left(\frac{\underset{2}{\Omega}}{n}g_{jk} - \underset{2}{\Omega}_{jk}\right) - c_{jk}\left(\frac{\underset{2}{\Omega}}{n}g_{il} - \underset{2}{\Omega}_{il}\right) - \frac{c}{n}\underset{2}{\Omega}_{il}g_{jk} + \frac{c}{n}\underset{2}{\Omega}_{jk}g_{il} = 0. \quad (3.12)$$

Hence

$$c_{il}\left(\frac{\underset{2}{\Omega}}{n}g_{jk} - \underset{2}{\Omega}_{jk}\right) - c_{jk}\left(\frac{\underset{2}{\Omega}}{n}g_{il} - \underset{2}{\Omega}_{il}\right) -$$

$$- \frac{c}{n}\underset{2}{\Omega}_{il}g_{jk} + \frac{c\,\underset{2}{\Omega}}{n\cdot n}g_{il}g_{jk} - \frac{c\,\underset{2}{\Omega}}{n\cdot n}g_{il}g_{jk} +$$

$$+ \frac{c}{n}\underset{2}{\Omega}_{jk}g_{il} - \frac{c\,\underset{2}{\Omega}}{n\cdot n}g_{il}g_{jk} + \frac{c\,\underset{2}{\Omega}}{n\cdot n}g_{il}g_{jk} = 0.$$

Grouping summands properly we get

$$\left(c_{il} - \frac{c}{n}g_{il}\right)\left(\frac{\underset{2}{\Omega}}{n}g_{jk} - \underset{2}{\Omega}_{jk}\right) - \left(c_{jk} - \frac{c}{n}g_{jk}\right)\left(\frac{\underset{2}{\Omega}}{n}g_{il} - \underset{2}{\Omega}_{il}\right) = 0.$$

As far as $c_{il} \neq \frac{c}{n} g_{il}$, we can substitute ξ^{ij} so, that

$$\left(c_{\alpha\beta} - \frac{c}{n} g_{\alpha\beta}\right) \xi^{\alpha\beta} = 1.$$

Multiplying (3.12) by ξ^{il}, and wrapping by i and l, we see that

$$\underset{2}{\Omega}_{ij} = \frac{\underset{2}{\Omega}}{n} g_{ij}.$$

Taking into account (3.9) we obtain that

$$P_{ij} = \sigma_{ij} - \frac{\underset{2}{\Omega}}{n} g_{ij} \tag{3.13}$$

Theorem is proved. □

Substituting (3.10) into (3.2) we get

$$\left(\Delta_1 \sigma - \frac{2\underset{2}{\Omega}}{n}\right)(c_{ik} g_{jl} - g_{jk} c_{il} + g_{il} c_{jk} - g_{ik} c_{jl}) = 0,$$

whence

$$\frac{2\underset{2}{\Omega}}{n} = \Delta_1 \sigma.$$

Therefore (3.13) can be written as follows:

$$P_{ij} = \sigma_{i,j} - \sigma_i \sigma_j - \tfrac{1}{2} \sigma_\alpha \sigma^\alpha g_{ij}, \tag{3.14}$$

whence its covariant derivative is

$$P_{ij,k} = \sigma_{i,jk} - \sigma_{i,k} \sigma_j - \sigma_i \sigma_{j,k} - \sigma_{\alpha,k} \sigma^\alpha g_{ij}.$$

Alternating by indices $(j\ k)$, and taking into account (3.5) and Ricci's identity, we obtain

$$\sigma_\alpha R^\alpha_{ijk} - \sigma_j \sigma_{i,k} - \sigma_k \sigma_{i,j} - \sigma_{\alpha,k} \sigma^\alpha g_{ij} + \sigma_{\alpha,j} \sigma^\alpha g_{ik} = 0,$$

$$\sigma_\alpha \sigma^\alpha_k g_{ij} - \sigma_\alpha \sigma^\alpha_j g_{ik} + \sigma_k \sigma_{ij} - \sigma_j \sigma_{ik} + \\ + \sigma_k \sigma_{i,j} - \sigma_j \sigma_{i,k} - \sigma_{\alpha,k} \sigma^\alpha g_{ij} + \sigma_{\alpha,j} \sigma^\alpha g_{ik} = 0.$$

Substituting (2.3) and grouping summands we get:

$$\sigma_k (2\sigma_{i,j} - \sigma_\alpha \sigma^\alpha g_{ij}) - \sigma_j (2\sigma_{i,k} - \sigma_\alpha \sigma^\alpha g_{ik}) = 0.$$

Since $\sigma_k \not\equiv 0$, there exists a vector ξ^k such that $\sigma_\alpha \xi^\alpha = 1$. Then

$$2\sigma_{i,j} - \sigma_\alpha \sigma^\alpha g_{ij} = \sigma_j \tau_i, \tag{3.15}$$

where

$$\tau_i = 2\sigma_{i,\alpha} \xi^\alpha - \sigma_\alpha \sigma^\alpha \xi_i.$$

Alternating we get that $\tau_i = \rho \sigma_i$, where $\rho = \tau_\alpha \xi^\alpha$.

Then equation (3.15) will be transformed into the following oneL

$$\sigma_{i,j} = \tfrac{1}{2}\rho\sigma_i\sigma_j + \Delta_1\sigma g_{ij}.$$

Hence one can write down (3.10) and (3.14) as follows:

$$P_{ij} = \left(\tfrac{1}{2}\rho - 1\right)\sigma_i\sigma_j + \tfrac{1}{2}\Delta_1\sigma g_{ij},$$

$$R_{hijk} = \left(\tfrac{1}{2}\rho - 1\right)(\sigma_h\sigma_k g_{ij} - \sigma_h\sigma_j g_{ik} + \sigma_i\sigma_j g_{hk} - \sigma_i\sigma_k g_{hj}) + \\ + \Delta_1\sigma(g_{hk}g_{ij} - g_{hj}g_{ik}).$$

It is easy to see that $\left(\frac{1}{2}\rho - 1\right) = f_1(\sigma)$ and $\Delta_1\sigma = f_2(\sigma)$.

An interesting group of conformally flat conformally reducible spaces are subprojective spaces. Subprojective spaces are also known as *Kagan's* spaces. Subprojective Kagan's spaces are characterized by the following identities, [13, 14]:

$$R_{ijkl} = Q(v)(g_{ik}v_{,j}v_{,l} + g_{jl}v_{,i}v_{,k} - g_{il}v_{,j}v_{,k} - g_{jk}v_{,i}v_{,l}) - \\ - 2S(v)(g_{il}g_{jk} - g_{ik}g_{jl}),$$

$$P_{jk} = S(v)g_{jk} + Q(v)v_{,j}v_{,k}.$$

Metric of subprojective space V_n can be reduced in a preselected coordinate system:

$$ds^2 = e^{-2\sigma(x^1)}(e_1 dx^{1^2} + e_2 dx^{2^2} + \ldots + e_n dx^{n^2}), \tag{3.16}$$

$$ds^2 = e^{-2\sigma(x)}(e_1 dx^{1^2} + e_2 dx^{2^2} + \ldots + e_n dx^{n^2}), \quad x = \sqrt{e_1 x^{1^2} + e_2 x^{2^2} + \ldots + e_n x^{n^2}}, \tag{3.17}$$

$$ds^2 = e^{-2\sigma(x^1)}(2dx^1 dx^2 + e_3 dx^{3^2} + \ldots + e_n dx^{n^2}). \tag{3.18}$$

From the structure of our metric it is easy to see that subprojective spaces are conformally reducible pseudo-Riemannian spaces. Moreover, a multiplier of conformality depends on the non-isotropic and therefore on isotropic coordinate x^1 of a flat space whose metric is presented in brackets. In the second case this multiplier depends solely on the distance to the coordinates axes.

Thus, we proved the theorem

Theorem 3.3. *Every conformal reducible space of the main type can be conformally flat space if and only if it is a subprojective space of Kagan.*

Expressions (3.16), (3.17), (3.18) give a complete classification of conformally flat conformal reducible pseudo-Riemannian spaces of the main type.

4. Conclusions

The presented paper studies conformal reducible spaces under assumption that they admit conformal mappings on flat spaces. It is proved that in this case they are subprojective spaces of Kagan. This allows to carry out an exhaustive classification of these spaces. It is demonstrated that conformally flat conformal reducible pseudo-Riemannian spaces are quasi-Einstein spaces. Results of the paper can be effectively applied for further research in mechanics, geometry, general relativity theory [17, 27] and topology [20]. In particular, under certain conditions, the obtained equations describe the state of an ideal fluid.

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