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Explicit formulae for Chern-Simons invariants of the hyperbolic $J(2n, -2m)$ knot orbifolds

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Abstract. We calculate the Chern-Simons invariants of the hyperbolic double twist knot orbifolds using the Schläfli formula for the generalized Chern-Simons function on the family of cone-manifold structures of double twist knots.

Анотація. В роботі обчислено інваріанти Черна-Саймонса гіперболічних орбіфолдів, що мають тип двічі скрученого вузла, використовуючи формулу Шлафлі для узагальненої функції Черна-Саймонса для сім'ї конусних структур двічі скручених вузлів.

1. Introduction

Chern-Simons invariants of hyperbolic knot orbifolds are computed explicitly for a few infinite families in [2–4] using the "Schläfli formula".

In this paper, we present the explicit formulae for Chern-Simons invariants of the hyperbolic double twist knot orbifolds and we present them numerically for some of double twist knot orbifolds. A brief history of Chern-Simons invariant can be found in [2–4]. A double twist knot is denoted by $C(2n, 2m)$ according to Conway notation or by $J(2n, -2m)$ according to Hoste-Shanahan notation. Figure 2 presents $C(2n, 2m)$ for $m, n > 0$.

For a two-bridge hyperbolic link, there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ for each link K such that the cone-manifold $K(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_0)$,

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Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$, [5,9,11,12]. We will use the Chern-Simons invariant of the lens space $L(4nm+1, 2n(2m-1)+1)$ calculated in [7]. The following theorem gives the Chern-Simons invariant formulae for the hyperbolic $C(2n, 2m)$ knots. Let $S_k(v)$ be the *Chebychev polynomials* defined by

$$S_0(v) = 1, \qquad S_1(v) = v,$$
$$S_k(v) = vS_{k-1}(v) - S_{k-2}(v),$$

for all integers k.

Theorem 1.1. *Let $X_{2n}^{2m}(\alpha)$ be the hyperbolic cone-manifold with underlying space S^3 and a singular set $C(2n, 2m)$ of cone-angle $0 \leqslant \alpha < \alpha_0$. Let k be a positive integer such that k-fold cyclic covering of $X_{2n}^{2m}\left(\frac{2\pi}{k}\right)$ is hyperbolic. Then the Chern-Simons invariant of $X_{2n}^{2m}\left(\frac{2\pi}{k}\right)$ (mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd) is given by the following formula:*

$$\begin{aligned}
\mathrm{cs}\Big(X_{2n}^{2m}\big(\tfrac{2\pi}{k}\big)\Big) &\equiv \frac{1}{2}\mathrm{cs}\Big(L(4nm+1, 2n(2m-1)+1)\Big) + \\
&+ \frac{1}{4\pi^2}\int_{\frac{2\pi}{k}}^{\alpha_0} \mathrm{Im}\left(2 * \log\left(-\frac{M^2(S_n(v)-S_{n-1}(v)) - (S_{n-1}(v)-S_{n-2}(v))}{(S_n(v)-S_{n-1}(v)) - M^2(S_{n-1}(v)-S_{n-2}(v))}\right)\right) d\alpha \\
&+ \frac{1}{4\pi^2}\int_{\alpha_0}^{\pi} \mathrm{Im}\left(\log\left(-\frac{M^2(S_n(v_1)-S_{n-1}(v_1)) - (S_{n-1}(v_1)-S_{n-2}(v_1))}{(S_n(v_1)-S_{n-1}(v_1)) - M^2(S_{n-1}(v_1)-S_{n-2}(v_1))}\right)\right) d\alpha \\
&+ \frac{1}{4\pi^2}\int_{\alpha_0}^{\pi} \mathrm{Im}\left(\log\left(-\frac{M^2(S_n(v_2)-S_{n-1}(v_2)) - (S_{n-1}(v_2)-S_{n-2}(v_2))}{(S_n(v_2)-S_{n-1}(v_2)) - M^2(S_{n-1}(v_2)-S_{n-2}(v_2))}\right)\right) d\alpha,
\end{aligned}$$

where for $M = e^{\frac{i\alpha}{2}}$, x, x_1, and x_2 are zeros of Riley-Mednykh polynomial $\phi_{2n}^{2m}(x, M)$ in Theorem 2.5. As α decreases to α_0 both x_1 and x_2 approach a common value x. One of x_1 and x_2 comes from the component of x, and the other comes from the component of $\bar{x}$. Moreover, v satisfies the inequality, [16, Lemma 3.9]:

$$\mathrm{Im}\Big((S_n(v) - S_{n-1}(v))\overline{\big(S_{n-1}(v) - S_{n-2}(v)\big)}\Big) \geqslant 0.$$

2. $C(2n, 2m)$ KNOTS

A general reference for this section is [8]. A knot with $2n$ right-handed vertical crossings and $2m$ left-handed horizontal crossings as in Figure 2.1 is $C(2n, 2m)$ knot according to Conway's notation. One can easily check that the slope of $C(2n, 2m)$ is $2m/(4nm+1)$ which is equivalent to the knot with slope $(2n(2m-1)+1)/(4nm+1)$ [14].

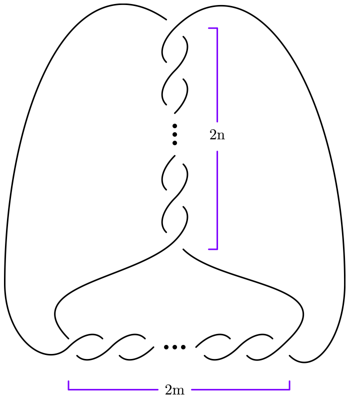

FIGURE 2.1. $C(2n, 2m)$

We will use the following presentation of the fundamental group of $C(2n, 2m)$ knot(equivalently, $J(2n, -2m)$ knot) in [8]. In [8], Hoste and Shanahan asked whether their presentation of the fundamental group for double twist knots can be derived from Schubert's canonical 2-bridge diagram or not. The following proposition can also be obtained by reading off the fundamental group from the Schubert normal form of $C(2n, 2m)$ with slope $2m/(4nm+1)$ [13,14] which answers Hoste-Shanahan's question completely for $C(2n, 2m)$ knots. Let X_{2n}^{2m} be $S^3 \backslash C(2n, 2m)$.

Proposition 2.1 ([4, Proposition2.2], [13,14])**.**

$$\pi_1(X_{2n}^{2m}) = \left\langle s, t \mid sw^m t^{-1} w^{-m} = 1 \right\rangle,$$

where $w = (t^{-1}s)^n (ts^{-1})^n$.

2.2. **The Chebychev polynomial.** Let $S_k(v)$ be the *Chebychev polynomials* defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_k(v) = vS_{k-1}(v) - S_{k-2}(v)$ for all integers k. The following explicit formula for $S_k(v)$ can be obtained by solving the above recurrence relation [17].

$$S_n(v) = \sum_{0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor} (-1)^i \binom{n-i}{i} v^{n-2i}$$

for $n \geqslant 0$, $S_n(v) = -S_{-n-2}(v)$ for $n \leqslant -2$, and $S_{-1}(v) = 0$. The following proposition 2.3 can be proved using the Cayley-Hamilton theorem [15].

Proposition 2.3 ([15, Proposition 2.4])**.** *Suppose* $V = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{C})$. *Then*

$$V^k = \begin{bmatrix} S_k(v) - dS_{k-1}(v) & bS_{k-1}(v) \\ cS_{k-1}(v) & S_k(v) - aS_{k-1}(v) \end{bmatrix}$$

where $v = \mathrm{tr}(V) = a + d$.

2.4. **The Riley-Mednykh polynomial.** Let

$$\rho(s) = \begin{bmatrix} M & 1 \\ 0 & M^{-1} \end{bmatrix}, \qquad \rho(t) = \begin{bmatrix} M & 0 \\ 2-v & M^{-1} \end{bmatrix},$$

and let

$$c = \begin{bmatrix} 0 & -(\sqrt{2-v}\,)^{-1} \\ \sqrt{2-v} & 0 \end{bmatrix}.$$

Then from the above Proposition 2.3, we get the following Theorem 2.5 which can be found in [16]. We include the proof for readers' convenience. Let $\rho(s) = S$, $\rho(t) = T$, and $\rho(w) = W$. Then $\mathrm{tr}(T^{-1}S) = v = \mathrm{tr}(TS^{-1})$. Let also $v = x + M^2 + M^{-2}$.

Theorem 2.5 ([16])**.** *ρ is a representation of $\pi_1(X_{2n}^{2m})$ if and only if x is a root of the following Riley-Mednykh polynomial,*

$$\phi_{2n}^{2m}(x, M) = S_m(z) + \left[-1 + xS_{n-1}(v)(S_n(v) + (1-v)S_{n-1}(v))\right] S_{m-1}(z).$$

Proof. Since

$$T^{-1}S = \begin{bmatrix} 1 & M^{-1} \\ M(M^{-2} + (x-2) + M^2) & M^{-2} + x - 1 + M^2 \end{bmatrix},$$

$$TS^{-1} = \begin{bmatrix} 1 & -M \\ -M^{-1}(M^{-2} + (x-2) + M^2) & M^{-2} + x - 1 + M^2 \end{bmatrix},$$

we have that

$$(T^{-1}S)^n = \begin{bmatrix} S_n(v) - (v-1)S_{n-1}(v) & M^{-1}S_{n-1}(v) \\ M(v-2)S_{n-1}(v) & S_n(v) - S_{n-1}(v) \end{bmatrix},$$

$$(TS^{-1})^n = \begin{bmatrix} S_n(v) - (v-1)S_{n-1}(v) & -MS_{n-1}(v) \\ -M^{-1}(v-2)S_{n-1}(v) & S_n(v) - S_{n-1}(v) \end{bmatrix}.$$

Hence

$$W = (T^{-1}S)^n(TS^{-1})^n = \begin{bmatrix} W_{11} & W_{12} \\ (2-v)W_{12} & W_{22} \end{bmatrix},$$

where

$$W_{11} = S_n^2(v) + (2-2v)S_n(v)S_{n-1}(v) + \\ + (1 + 2M^{-2} - 2v - M^{-2}v + v^2)S_{n-1}^2(v),$$

$$W_{12} = (M^{-1} - M)S_n(v)S_{n-1}(v) + (Mv - M - M^{-1})S_{n-1}^2(v),$$

$$W_{22} = S_n^2(v) - 2S_n(v)S_{n-1}(v) + (1 + 2M^2 - M^2v)S_{n-1}^2(v).$$

Let $z = \mathrm{tr}(W)$. Then, since $S_n^2(v) - vS_n(v)S_{n-1}(v) + S_{n-1}^2(v) = 1$ (by [18, Lemma 2.1] or by induction),

$$\begin{aligned} z &= W_{11} + W_{22} = \\ &= 2\big(S_n^2(v) - vS_n(v)S_{n-1}(v) + S_{n-1}^2(v)\big) + \\ &\qquad\qquad + \left(2M^{-2} + 2M^2 - 2v - M^{-2}v - M^2v + v^2\right) S_{n-1}^2(v) \\ &= 2 + (v-2)xS_{n-1}^2(v). \end{aligned}$$

By Proposition 2.3, we have

$$(W)^m = \begin{bmatrix} S_m(z) - W_{22}S_{m-1}(z) & W_{12}S_{m-1}(z) \\ (2-v)W_{12}S_{m-1}(z) & S_m(z) - W_{11}S_{m-1}(z) \end{bmatrix}.$$

Therefore

$$\mathrm{tr}(SW^m c)/\sqrt{2-v} = \big(\left(M - M^{-1}\right) W_{12}S_{m-1}(z) + S_m(z) - W_{11}S_{m-1}(z)\big)$$

gives $\phi_{2n}^{2m}(x, M)$, [4]. □

3. Longitude

Let $l = w^m(w^*)^m$, where w^* is the word obtained by reversing w. Then l is the longitude which is null-homologous in X_{2n}^{2m}. Recall that $\rho(w) = W$. Let also $\widetilde{W} = \rho(w^*)$. Then it is easy to see that $\widetilde{W}$ can be written as

$$\widetilde{W} = \begin{bmatrix} \tilde{W}_{22} & \tilde{W}_{12} \\ (2-v)\tilde{W}_{12} & \tilde{W}_{11} \end{bmatrix},$$

where $\tilde{W}_{ij}$ is obtained by W_{ij} by replacing M with M^{-1}. Similar computation was introduced in [8]. Hence,

$$\begin{aligned} \tilde{W}_{11} &= S_n^2(v) + (2-2v)\, S_n(v)S_{n-1}(v) + \\ &\qquad\qquad + (1 + 2M^2 - 2v - M^2v + v^2)\, S_{n-1}^2(v), \\ \tilde{W}_{12} &= (M - M^{-1})\, S_n(v)S_{n-1}(v) + (M^{-1}v - M^{-1} - M)\, S_{n-1}^2(v), \\ \tilde{W}_{22} &= S_n^2(v) - 2S_n(v)S_{n-1}(v) + (1 + 2M^{-2} - M^{-2}v)\, S_{n-1}^2(v). \end{aligned}$$

The following lemma was introduced in [8] with slightly different coordinates. Let $L = \rho(l)_{11}$ be the left upper entry of $\rho(l)$.

Lemma 3.1 ([8]). *$W_{21}L + \tilde{W}_{21} = 0$.*

Theorem 3.2.

$$L = -\frac{M^2\left(S_n(v) - S_{n-1}(v)\right) - \left(S_{n-1}(v) - S_{n-2}(v)\right)}{\left(S_n(v) - S_{n-1}(v)\right) - M^2\left(S_{n-1}(v) - S_{n-2}(v)\right)}.$$

Proof. A direct computation shows that $W_{21}L + \tilde{W}_{21} = 0$ in Lemma 3.1. Now theorem follows by substituting $S_n(v) + S_{n-2}(v)$ for $vS_{n-1}(v)$. □

4. Schläfli formula for the generalized Chern-Simons function

The general references for this section are [2,3,6,7,10,19] and [4].

In [6], Hilden, Lozano, and Montesinos-Amilibia defined the *generalized Chern-Simons function* on the oriented cone-manifold structures which matches up with the Chern-Simons invariant when the cone-manifold is the Riemannian manifold.

Below, we briefly introduce the generalized Chern-Simons function on the family of $C(2n, 2m)$ cone-manifold structures. For an oriented knot $C(2n, 2m)$, we orient its chosen meridian s such that the orientation of s followed by the orientation of $C(2n, 2m)$ coincides with orientation of $\mathbb{S}^3$. Here, we use the definition of the lens space in [7] so that we can have the right orientation when it is combined with the following frame field.

On the Riemannian manifold

$$\mathbb{S}^3 - C(2n, 2m) - s$$

we choose a *special* frame field $\Gamma = (e_1, e_2, e_3)$ which is an orthonormal frame field such that for each point x near $C(2n, 2m)$, $e_1(x)$ has the direction given by knot's orientation, $e_2(x)$ has the tangent direction of the meridian curve, and $e_3(x)$ has the knot to point x direction. Such a special frame field always exists by [6, Proposition 3.1]. From Γ we obtain an orthonormal frame field Γ_α on $X_{2n}^{2m}(\alpha) - s$ by the Gram-Schmidt orthogonalization process with respect to the Riemannian structure of the cone manifold $X_{2n}^{2m}(\alpha)$. Moreover, it can be made special by deforming it in a neighborhood of the singular set $C(2n, 2m)$ and s, if necessary. Thus, Γ' is an extension of Γ to $S^3 - C(2n, 2m)$. To the cone-manifold $X_{2n}^{2m}(\alpha)$, we assign the following real number

$$I\left(X_{2n}^{2m}(\alpha)\right) = \frac{1}{2} \int\limits_{\Gamma(S^3 - T_{2n} - s)} Q - \frac{1}{4\pi}\tau(s, \Gamma') - \frac{1}{4\pi}\left(\frac{\beta\alpha}{2\pi}\right),$$

where $-2\pi \leqslant \beta \leqslant 2\pi$ is the angle of the lifted holonomy of the singular locus of $X_{2n}^{2m}(\alpha)$, Q is the Chern-Simons form:

$$Q = \frac{1}{4\pi^2}\left(\theta_{12} \wedge \theta_{13} \wedge \theta_{23} + \theta_{12} \wedge \Omega_{12} + \theta_{13} \wedge \Omega_{13} + \theta_{23} \wedge \Omega_{23}\right),$$

and

$$\tau(s, \Gamma') = -\int_{\Gamma'(s)} \theta_{23},$$

where (θ_{ij}) is the connection 1-form, (Ω_{ij}) is the curvature 2-form of the Riemannian connection on $X_{2n}^{2m}(\alpha)$ and the integral is over the orthonormalizations of the same frame field. When $\alpha = \frac{2\pi}{k}$ for some positive integer, $I\left(X_{2n}^{2m}\left(\frac{2\pi}{k}\right)\right)$ (mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd) is independent of the frame field Γ and of the representative in the equivalence class $\overline{\beta}$ and hence becomes an invariant of the orbifold $X_{2n}^{2m}\left(\frac{2\pi}{k}\right)$. The quantity $I\left(X_{2n}^{2m}\left(\frac{2\pi}{k}\right)\right)$ (mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd) is called *the Chern-Simons invariant of the orbifold* and is denoted by $\mathrm{cs}\left(X_{2n}^{2m}\left(\frac{2\pi}{k}\right)\right)$.

We have the following "Schläfli formula" for the generalized Chern-Simons function on the family of $C(2n, 2m)$ cone-manifold structures.

Theorem 4.1 ([7, Theorem 1.2])**.** *For a family of geometric cone-manifold structures, $X_{2n}^{2m}(\alpha)$, and differentiable functions $\alpha(t)$ and $\beta(t)$ of t we have*

$$dI\left(X_{2n}^{2m}(\alpha)\right) = -\tfrac{1}{4\pi^2}\,\beta d\alpha.$$

5. Proof of Theorem 1.1

For $n \geqslant 1$ and $M = e^{i\frac{\alpha}{2}}$, $\phi_{2n}^{2m}(x, M)$ have $2mn$ component zeros. The component passing through

$$(x_1, x_2) = \left(2 - 2\cos\left(\frac{\pi(2m+1)}{4nm+1}\right), 2 - 2\cos\left(\frac{\pi(2m-1)}{4nm+1}\right)\right)$$

at $\alpha = \pi$ is the geometric component by [7, Theorem 2.1]. Note that $2 - x_1 > 0$ and $2 - x_2 > 0$. For each $C(2n, 2m)$, there exists an angle $\alpha_0 \in [\frac{2\pi}{3}, \pi)$ such that $C(2n, 2m)$ is hyperbolic for $\alpha \in (0, \alpha_0)$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha \in (\alpha_0, \pi]$, [5, 9, 11, 12]. Denote by $D(X_{2n}^{2m}(\alpha))$ the set of zeros of the discriminant of $\phi_{2n}^{2m}(x, e^{i\frac{\alpha}{2}})$ over x. Then α_0 will be one of $D(X_{2n}^{2m}(\alpha))$.

On the geometric component we can calculate the Chern-Simons invariant of an orbifold $X_{2n}^{2m}\left(\frac{2\pi}{k}\right)$ mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd), where k is a positive integer such that k-fold cyclic covering of $X_{2n}^{2m}\left(\frac{2\pi}{k}\right)$ is hyperbolic:

$$\begin{aligned}
&\mathrm{cs}\Big(X_{2n}^{2m}\left(\tfrac{2\pi}{k}\right)\Big) \equiv I\Big(X_{2n}^{2m}\left(\tfrac{2\pi}{k}\right)\Big) \ \left(\mathrm{mod}\, \tfrac{1}{k}\right)\\
&\quad \equiv I\Big(X_{2n}^{2m}(\pi)\Big) + \tfrac{1}{4\pi^2}\int_{\frac{2\pi}{k}}^{\pi} \beta\, d\alpha \ \left(\mathrm{mod}\, \tfrac{1}{k}\right)\\
&\quad \equiv \frac{1}{2}\mathrm{cs}\Big(L(4nm+1, 2n(2m-1)+1)\Big) +\\
&\qquad + \frac{1}{4\pi^2}\int_{\frac{2\pi}{k}}^{\alpha_0} \mathrm{Im}\Bigg(2 * \log\bigg(-\frac{M^2(S_n(v) - S_{n-1}(v)) - (S_{n-1}(v) - S_{n-2}(v))}{(S_n(v) - S_{n-1}(v)) - M^2(S_{n-1}(v) - S_{n-2}(v))}\bigg)\Bigg)\, d\alpha
\end{aligned}$$

$$+\frac{1}{4\pi^2}\int_{\alpha_0}^{\pi}\mathrm{Im}\Bigg(\log\Bigg(-\frac{M^2(S_n(v_1)-S_{n-1}(v_1))-(S_{n-1}(v_1)-S_{n-2}(v_1))}{(S_n(v_1)-S_{n-1}(v_1))-M^2(S_{n-1}(v_1)-S_{n-2}(v_1))}\Bigg)\Bigg)\,d\alpha$$

$$+\frac{1}{4\pi^2}\int_{\alpha_0}^{\pi}\mathrm{Im}\Bigg(\log\Bigg(-\frac{M^2(S_n(v_2)-S_{n-1}(v_2))-(S_{n-1}(v_2)-S_{n-2}(v_2))}{(S_n(v_2)-S_{n-1}(v_2))-M^2(S_{n-1}(v_2)-S_{n-2}(v_2))}\Bigg)\Bigg)\,d\alpha,$$

(mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd),

where the second equivalence comes from Theorem 4.1 and the third equivalence comes from the fact that

$$I\big(X_{2n}^{2m}(\pi)\big)\equiv\tfrac{1}{2}\mathrm{cs}\big(L(4nm+1,2n(2m-1)+1)\big)\ (\mathrm{mod}\tfrac{1}{2}),$$

from Theorem 3.2, and from geometric interpretations of hyperbolic and spherical holonomy representations.

The following theorem gives the Chern-Simons invariant of the Lens space $L(4nm+1,2n(2m-1)+1)$.

Theorem 5.1 ([7, Theorem 1.3])**.**

$$\mathrm{cs}\big(L(4nm+1,2n(2m-1)+1)\big)\equiv\frac{m-n}{4nm+1}\ (\mathrm{mod}1).$$

6. Chern-Simons invariants of the hyperbolic $C(2n,2m)$ knot orbifolds and of its cyclic coverings

Table 6.1 gives the approximate Chern-Simons invariant of $C(2n,2m)$ for n between 1 and 4, m between 1 and 4 with $n\geqslant m$. Since $C(2,2)$, $C(4,4)$, $C(6,6)$, $C(8,8)$ are amphicheiral knots, their Chern-Simons invariants are *zero* as expected. We used Simpson's rule for the approximation with 2×10^4 (10^4 in Simpson's rule) intervals from 0 to α_0 and 2×10^4 (10^4 in Simpson's rule) intervals from α_0 to π. Table 6.2 gives the approximate Chern-Simons invariant of the hyperbolic orbifold, $\mathrm{cs}\big(X_{2n}^{2m}(\frac{2\pi}{k})\big)$ for n between 1 and 4, m between 1 and 4 with $n\geqslant m$, and for k between 3 and 10, and of its cyclic covering, $\mathrm{cs}\big(M_k(X_{2n}^{2m})\big)$ except amphicheiral knots. We used Simpson's rule for the approximation with 2×10^2 (10^2 in Simpson's rule) intervals from $2\pi/k$ to α_0 and 2×10^2 (10^2 in Simpson's rule) intervals from α_0 to π.

We used Mathematica for the calculations. We record here that our data in Table 6.1 and those obtained from SnapPy [1] match up up to existing decimal points and our data in Table 6.2. For computational reasons, we need α_0 being the bifurcation point of the geometric solution of the Riley-Mednykh polynomial as described in Theorem 1.1.

Table 6.1. Chern-Simons invariant of X_{2n}^{2m} for n between 1 and 4, m between 1 and 4 with $n \geqslant m$ except amphicheiral knots:

$2n$	$2m$	α_0	$\mathrm{cs}\big(X_{2n}^{2m}\big)$
2	2	2.094395102393195	0
4	2	2.574140778131840	0.34402298
6	2	2.750685152010280	0.27786688
8	2	2.843209532683532	0.24222232
4	4	2.847642272262783	0
6	4	2.942465754372979	0.42782933
8	4	2.990939179603150	0.38923730
6	6	3.007517657179940	0
8	6	3.040474611156828	0.46103929
8	8	3.065453796328835	0

Table 6.2. Chern-Simons invariant of the hyperbolic orbifold, $\mathrm{cs}\big(X_{2n}^{2m}(\frac{2\pi}{k})\big)$ (mod $\frac{1}{k}$ if k is even or mod $\frac{1}{2k}$ if k is odd) for n between 1 and 4, m between 1 and 4 with $n \geqslant m$, and for k between 3 and 10, and of its cyclic covering, $\mathrm{cs}\big(M_k(X_{2n}^{2m})\big)$ except amphicheiral knots:

k	$\mathrm{cs}\big(X_4^2(\frac{2\pi}{k})\big)$	$\mathrm{cs}\big(M_k(X_4^2)\big)$
3	0.0875301	0.26259
4	0.144925	0.579699
5	0.0784576	0.392288
6	0.0351571	0.210943
7	0.00506505	0.0354553
8	0.108039	0.864313
9	0.0218112	0.196301
10	0.0530574	0.530574

k	$\mathrm{cs}\big(X_6^2(\frac{2\pi}{k})\big)$	$\mathrm{cs}\big(M_k(X_6^2)\big)$
3	0.0449535	0.13486
4	0.0876043	0.350417
5	0.0165337	0.0826684
6	0.138167	0.829004
7	0.0120078	0.0840545
8	0.0430876	0.3447
9	0.012125	0.109125
10	0.0876213	0.876213

k	$\mathrm{cs}\big(X_8^2(\frac{2\pi}{k})\big)$	$\mathrm{cs}\big(M_k(X_8^2)\big)$
3	0.0161266	0.0483799
4	0.0536832	0.214733
5	0.0817026	0.408513
6	0.103012	0.618074
7	0.0481239	0.336867
8	0.00768503	0.0614802
9	0.032221	0.289989
10	0.0521232	0.521232

k	$\mathrm{cs}\big(X_6^4(\frac{2\pi}{k})\big)$	$\mathrm{cs}\big(M_k(X_6^4)\big)$
3	0.125912	0.377736
4	0.192764	0.771058
5	0.0360431	0.180216
6	0.0996796	0.598077
7	0.00284328	0.0199029
8	0.0554674	0.443739
9	0.0409685	0.368717
10	0.0294401	0.294401

k	$\mathrm{cs}\big(X_8^4(\frac{2\pi}{k})\big)$	$\mathrm{cs}\big(M_k(X_8^4)\big)$
3	0.098074	0.294222
4	0.157843	0.631371
5	0.0993608	0.496804
6	0.0622858	0.373715
7	0.0365103	0.255572
8	0.0174882	0.139906
9	0.00284881	0.0256393
10	0.091224	0.91224

k	$\mathrm{cs}\big(X_8^6(\frac{2\pi}{k})\big)$	$\mathrm{cs}\big(M_k(X_8^6)\big)$
3	0.138854	0.416562
4	0.214725	0.858898
5	0.0628859	0.31443
6	0.128841	0.773046
7	0.0332457	0.23272
8	0.0866094	0.692875
9	0.0170324	0.153291
10	0.0613865	0.613865

Note that the Chern-Simons invariant of the hyperbolic orbifold

$$\mathrm{cs}\big(X_{2n}^{2m}(\tfrac{2\pi}{k})\big)$$

is only defined modulo $\frac{1}{k}$ [7, Theorem 1.4] and we only get modulo $\frac{1}{2k}$ for k odd [7, Theorem 1.4].

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