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On the asymptotic behavior at infinity of one mapping class

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Dedicated to the memory of Professor Oleksandr Bakhtin

Abstract. We study the asymptotic behavior at infinity of ring Q-homeomorphisms with respect to p-modulus for $p > n$.

Анотація. Ми досліджуємо асимптотичну поведінку на нескінченності кільцевих Q-гомеоморфізмів відносно p-модуля для $p > n$.

1. Introduction

Let us recall some definitions, see [36]. Let Γ be a path family in $\mathbb{R}^n$, $n \geqslant 2$. A Borel function $\rho\colon \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, (abbr. $\rho \in \mathrm{adm}\Gamma$), if

$$\int\limits_{\gamma} \rho(x)ds \ \geqslant \ 1$$

for all locally rectifiable $\gamma \in \Gamma$.

Let $p \in (1, \infty)$. Recall also that the *p–modulus* of Γ is the quantity

$$\mathrm{M}_p(\Gamma) \ = \ \inf_{\rho \in \mathrm{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)dm(x),$$

where $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$.

Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, $x_0 \in D$ and $d_0 = \mathrm{dist}(x_0, \partial D)$. For arbitrary sets E, F and G of $\mathbb{R}^n$ we denote by $\Delta(E, F, G)$ a set of all

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continuous curves $\gamma\colon [a,b] \to \mathbb{R}^n$ that connect E and F in G, i.e., such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$. Set

$$\begin{aligned} \mathbb{A}(x_0, r_1, r_2) =& \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}, \\ S_i = S(x_0, r_i) =& \{x \in \mathbb{R}^n : |x - x_0| = r_i\},\ i = 1, 2. \end{aligned}$$

Let $Q\colon D \to [0, \infty]$ be a measurable function. Recall that a homeomorphism $f\colon D \to \mathbb{R}^n$ is said to be a *Q-homeomorphism with respect to p-modulus* if

$$\mathrm{M}_p(f\Gamma) \leqslant \int\limits_D Q(x)\rho^p(x)\ dm(x) \tag{1.1}$$

for every family Γ of paths in D and every admissible function ρ for Γ.

This conception is a natural generalization of the geometric definition of a quasiconformal mapping: if $Q(x) \leqslant K < \infty$ a.e., then

- f is quasiconformal for $p = 2$ in $\mathbb{C}$, (see [1, Definition A, p. 21-22]) and for $p = n$ in $\mathbb{R}^n$, $n \geqslant 2$, ([36, 13.1 & 34.6]);
- f has local Lipschitz property, for $n - 1 < p < n$, f^{-1} is Lipschitz, for $p > n$, and the bounds for p are sharp (see [5]).

Note that the estimate of the type (1.1) was first established in the classical quasiconformal theory, (see [13, p. 221]). Further, it was obtained in [2, Lemma 2.1], for quasiconformal mappings in space $\mathbb{R}^n$, $n \geqslant 2$. This class of Q-homeomorphisms with respect to the n-modulus was first considered in the papers [16–18], see also the monograph [19].

The main goal of the theory of Q-homeomorphisms is to clarify various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. In particular, the problem of the local and boundary behavior of Q-homeomorphisms has been studied in $\mathbb{R}^n$ first in the case $Q \in BMO$ (bounded mean oscillation) in the papers [17, 18] and then in the case of $Q \in FMO$ (finite mean oscillation) and other cases in the papers [9, 10, 21, 24].

The following concept generalizes and localizes the concept of a Q-homeomorphism. It is motivated by the ring definition of quasiconformal mappings in the sense of Gehring (see [4]), introduced originally by Ryazanov, Srebro, and Yakubov on the plane, and later extended by Ryazanov and Sevost'yanov in the space $\mathbb{R}^n, n \geqslant 2$, (see [22], [19, Chapters VII and XI]).

Let $Q\colon D \to [0, \infty]$ be Lebesgue measurable function. We say that a homeomorphism $f\colon D \to \mathbb{R}^n$ is a *ring Q-homeomorphism with respect to p-modulus at $x_0 \in D$* if the relation

$$\mathrm{M}_p\big(\Delta(fS_1, fS_2, fD)\big) \ \leqslant \int\limits_{\mathbb{A}} Q(x)\eta^p(|x - x_0|)dm(x)$$

holds for any ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0$, $d_0 = \mathrm{dist}(x_0, \partial D)$, and for any measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)dr = 1.$$

The theory of ring Q-homeomorphisms for $p = n$ was studied in [19, 22, 23, 34], for $1 < p < n$ in [6–8, 25–27] and for $p > n$ in [11, 28–33]. This theory can be applied to the mappings of finite distortion belonging to the Orlicz–Sobolev classes $W^{1,\varphi}_{\mathrm{loc}}$ under the Calderon condition, and, in particular, to the Sobolev classes $W^{1,p}_{\mathrm{loc}}$ with $p > n - 1$, (see [8, 12]).

In this paper, we obtain an analogue of Martio-Rickman-Vaisal's theorem on the growth at infinity of quasi-regular mappings (see [15]). The case $p = n$ was studied in [34].

Denote by ω_{n-1} the area of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ in $\mathbb{R}^n$ and by

$$q_{x_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{S(x_0, r)} Q(x) d\mathcal{A}$$

the integral mean over the sphere $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, here $d\mathcal{A}$ is the element of the surface area.

Now we formulate a criterion which guarantees that a homeomorphism is a ring Q-homeomorphism with respect to p-modulus for $p > 1$ in $\mathbb{R}^n$, $n \geqslant 2$.

Proposition 1.1. [26, Theorem 2.3] *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and $Q\colon D \to [0, \infty]$ be a Lebesgue measurable function such that $q_{x_0}(r) \neq \infty$ for a.e. $r \in (0, d_0)$, $d_0 = \mathrm{dist}(x_0, \partial D)$. A homeomorphism $f\colon D \to \mathbb{R}^n$ is a ring Q-homeomorphism with respect to p-modulus at a point $x_0 \in D$ if and only if the inequality*

$$\mathrm{M}_p\big(\Delta(fS_1, fS_2, f\mathbb{A})\big) \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}\right)^{p-1}}$$

holds for any $0 < r_1 < r_2 < d_0$.

Following [14], a pair $\mathcal{E} = (A, C)$, where $A \subset \mathbb{R}^n$ is an open set and C is a nonempty compact set contained in A, is called *condenser*. We say that a condenser $\mathcal{E} = (A, C)$ *lies in a domain* D if $A \subset D$.

Clearly, if $f\colon D \to \mathbb{R}^n$ is a homeomorphism and $\mathcal{E} = (A, C)$ is a condenser in D then $(f(A), f(C))$ is also condenser in $f(D)$. Further, we denote $f(\mathcal{E}) = (f(A), f(C))$.

We will recall also the notion of the p-capacity. Given a condenser

$$\mathcal{E} = (A, C)$$

denote by $\mathcal{C}_0(A)$ the set of all continuous functions $u\colon A \to \mathbb{R}^1$ with compact support. Let also $\mathcal{W}_0(\mathcal{E}) = \mathcal{W}_0(A, C)$ be the family of nonnegative functions $u\colon A \to \mathbb{R}^1$ such that

(1) $u \in \mathcal{C}_0(A)$,

(2) $u(x) \geqslant 1$ for $x \in C$,

(3) u belongs to the class ACL and $|\nabla u| = \left(\sum\limits_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2\right)^{\frac{1}{2}}$.

Then for $p \geqslant 1$ the quantity

$$\mathrm{cap}_p \mathcal{E} = \mathrm{cap}_p(A, C) = \inf_{u \in \mathcal{W}_0(\mathcal{E})} \int\limits_A |\nabla u|^p dm(x)$$

is called the *p-capacity* of the condenser $\mathcal{E}$.

It is known ([35, Theorem 1]) that for $p > 1$

$$\mathrm{cap}_p \mathcal{E} = \mathrm{M}_p\big(\Delta(\partial A, \partial C, A \backslash C)\big). \tag{1.2}$$

Also for $p > n$ we have the another inequality

$$\mathrm{cap}_p(A, C) \geqslant n\Omega_n^{\frac{p}{n}} \left(\frac{p-n}{p-1}\right)^{p-1} \left[m^{\frac{p-n}{n(p-1)}}(A) - m^{\frac{p-n}{n(p-1)}}(C)\right]^{1-p}, \tag{1.3}$$

where Ω_n is a volume of the unit ball in $\mathbb{R}^n$ (see, e.g., (8.7) in [20]).

Let us recall the so-called isodiametric inequality or Bieberbach inequality (1915), see [3, Corollary 2.10.33]. Here and in what follows, $\mathrm{diam}(\cdot)$ denotes the Euclidean diameter in $\mathbb{R}^n, n \geqslant 2$.

Proposition 1.2. *Let E be a compact set in $\mathbb{R}^n, n \geqslant 2$. Then*

$$m(E) \leqslant 2^{-n} \Omega_n \left(\mathrm{diam} E\right)^n,$$

where Ω_n is a volume of the unit ball in $\mathbb{R}^n$.

2. Main results

Now we present the main result of our paper concerning the behavior at infinity of ring Q-homeomorphisms with respect to p-modulus for $p > n$.

Theorem 2.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a ring Q-homeomorphism with respect to p-modulus at a point x_0 with $p > n$, where x_0 is some point in $\mathbb{R}^n$. Suppose that for some numbers $r_0 > 1$ and $\kappa = \kappa(x_0) > 0$ the following condition*

$$q_{x_0}(t) \leqslant \kappa t^{p-n} \left(\ln t\right)^{\alpha} \tag{2.1}$$

holds for a.e. $t \in [r_0, +\infty)$. *If* $\alpha \in [0, p-1)$, *then*

$$\liminf_{R\to\infty} \frac{\mathrm{diam} f(B(x_0,R))}{(\ln R)^{\frac{p-\alpha-1}{p-n}}} \geqslant 2\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-\alpha-1}\right)^{\frac{p-1}{p-n}} > 0.$$

If $\alpha = p-1$, *then*

$$\liminf_{R\to\infty} \frac{\mathrm{diam} f(B(x_0,R))}{(\ln\ln R)^{\frac{p-1}{p-n}}} \geqslant 2\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-1}\right)^{\frac{p-1}{p-n}} > 0,$$

where $B(x_0,R) = \{x \in \mathbb{R}^n : |x-x_0| \leqslant R\}$.

Proof. Consider a condenser $\mathcal{E} = (A, C)$ in $\mathbb{R}^n$, where

$$A = \{x \in \mathbb{R}^n : |x-x_0| < R\},$$
$$C = \{x \in \mathbb{R}^n : |x-x_0| \leqslant r_0\},$$

and $1 < r_0 < R < \infty$. Then $f(\mathcal{E}) = (f(A), f(C))$ is a ringlike condenser in $\mathbb{R}^n$ and by (1.2), we have the equality

$$\mathrm{cap}_p f(\mathcal{E}) = \mathrm{M}_p\Big(\Delta\big(\partial f(A),\ \partial f(C),\ f(A\backslash C)\big)\Big).$$

Then due to (1.3) we have that

$$\mathrm{cap}_p(f(A), f(C)) \geqslant n\Omega_n^{\frac{p}{n}} \left(\tfrac{p-n}{p-1}\right)^{p-1} \left[m^{\frac{p-n}{n(p-1)}}(f(A)) - m^{\frac{p-n}{n(p-1)}}(f(C))\right]^{1-p},$$

whence

$$\mathrm{cap}_p(f(A), f(C)) \geqslant n\Omega_n^{\frac{p}{n}} \left(\frac{p-n}{p-1}\right)^{p-1} [m(f(A))]^{\frac{n-p}{n}}. \tag{2.2}$$

On the other hand, by Proposition 1.1, one gets

$$\mathrm{cap}_p(f(A), f(C)) \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{p-1}}. \tag{2.3}$$

Combining the inequalities (2.2) and (2.3), we obtain

$$n\Omega_n^{\frac{p}{n}} \left(\frac{p-n}{p-1}\right)^{p-1} [m(f(A))]^{\frac{n-p}{n}} \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{p-1}}.$$

Now, due to $\omega_{n-1} = n\Omega_n$, the last inequality can be rewritten as follows:

$$\Omega_n^{\frac{p}{n}-1} \left(\frac{p-n}{p-1}\right)^{p-1} [m(f(A))]^{\frac{n-p}{n}} \leqslant \left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{1-p}. \tag{2.4}$$

Consider the case when $\alpha \in [0, p-1)$. Then the following estimate follows from the condition (2.1):

$$\Omega_n^{\frac{p}{n}-1}\left(\frac{p-n}{p-1}\right)^{p-1}\left[m(f(A))\right]^{\frac{n-p}{n}} \leqslant$$
$$\leqslant \kappa\left(\frac{p-\alpha-1}{p-1}\right)^{p-1}\left((\ln R)^{\frac{p-\alpha-1}{p-1}}-(\ln r_0)^{\frac{p-\alpha-1}{p-1}}\right)^{1-p}.$$

Therefore,

$$m(f(A)) \geqslant$$
$$\geqslant \Omega_n \kappa^{\frac{n}{n-p}}\left(\frac{p-n}{p-\alpha-1}\right)^{\frac{n(p-1)}{p-n}}\left((\ln R)^{\frac{p-\alpha-1}{p-1}}-(\ln r_0)^{\frac{p-\alpha-1}{p-1}}\right)^{\frac{n(p-1)}{p-n}}.$$

Hence, by Proposition 1.2, we have

$$\operatorname{diam} f(B(x_0, R)) \geqslant$$
$$\geqslant 2\kappa^{\frac{1}{n-p}}\left(\frac{p-n}{p-\alpha-1}\right)^{\frac{p-1}{p-n}}\left((\ln R)^{\frac{p-\alpha-1}{p-1}}-(\ln r_0)^{\frac{p-\alpha-1}{p-1}}\right)^{\frac{p-1}{p-n}}.$$

Dividing the last inequality by $(\ln R)^{\frac{p-\alpha-1}{p-n}}$ and taking the lower limit for $R \to \infty$, we conclude

$$\liminf_{R\to\infty}\frac{\operatorname{diam} f(B(x_0, R))}{(\ln R)^{\frac{p-\alpha-1}{p-n}}} \geqslant 2\kappa^{\frac{1}{n-p}}\left(\frac{p-n}{p-\alpha-1}\right)^{\frac{p-1}{p-n}}.$$

Now we consider the case when $\alpha = p-1$. Then it follows from (2.4) that

$$\Omega_n^{\frac{p}{n}-1}\left(\frac{p-n}{p-1}\right)^{p-1}\left[m(f(A))\right]^{\frac{n-p}{n}} \leqslant \kappa\left(\ln\frac{\ln R}{\ln r_0}\right)^{1-p}.$$

Therefore,

$$m\big(f(B(x_0, R))\big) \geqslant \Omega_n \kappa^{\frac{n}{n-p}}\left(\frac{p-n}{p-1}\right)^{\frac{n(p-1)}{p-n}}\left(\ln\frac{\ln R}{\ln r_0}\right)^{\frac{n(p-1)}{p-n}}.$$

Hence, by Proposition 1.2, we obtain

$$\operatorname{diam} f(B(x_0, R)) \geqslant 2\kappa^{\frac{1}{n-p}}\left(\frac{p-n}{p-1}\right)^{\frac{p-1}{p-n}}\left(\ln\frac{\ln R}{\ln r_0}\right)^{\frac{p-1}{p-n}}.$$

Finally, dividing the last inequality by $(\ln\ln R)^{\frac{p-1}{p-n}}$ and taking the lower limit for $R\to\infty$, we conclude

$$\liminf_{R\to\infty}\frac{\operatorname{diam} f(B(x_0,R))}{(\ln\ln R)^{\frac{p-1}{p-n}}}\geqslant 2\kappa^{\frac{1}{n-p}}\left(\frac{p-n}{p-1}\right)^{\frac{p-1}{p-n}}.$$

This completes the proof. □

Remark 2.2. It is easy to construct examples of ring Q-homeomorphisms with respect to p-modulus on which the obtained estimates are attained.

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