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STUDY OF THE PROCESS OF SEEKING GLOBAL EXTREMUM OF FUNCTION BY SYMMETRIC ALGORITHMS WITH PARALLEL SPACE

Abstract. Theoretical and experimental research of seeking global extremum of function based on the application of the concept of symmetry is carried out.
Keywords: extremum of functions, methods of symmetry, gradient methods of optimization.

Problem statement

Implementation of a real-time optimization, is usually accompanied by various kinds of energy processes. Processes of energy flow and transformation are rather inertial.

The latter processes result in undesirable dynamic processes during the extremum seeking. Therefore, there is an important problem of providing the relevant power level of optimization process realization by the outflow of excess energy or by inflow of additional energy.

Analysis of recent research of the problem

To solve dynamic optimization problems, the relaxation method is often used [1,2]. It is implemented with the help of non-steady processes described by the vector differential equation:

$$\frac{dx}{dt} + k\ grad\ f(x) = 0, \quad k > 0; \tag{1}$$

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + grad f(x) = 0, \quad m > 0, \quad r > 0. \tag{2}$$

These processes with time converge (are established) to solve the given problem.

The relaxation method implemented according to equation (1) is called the steepest gradient descent method, and the relaxation method according to equation (2) is the heavy ball method. The former relaxation methods, as a rule, is referred to as a method of the local extremum

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seeking, and the latter (with the appropriate choice of parameters m and r) - to the methods of global extremum seeking.

It is known that the elongation of the surface of the goal function along one of the directions reduces the efficiency of the above mentioned relaxation methods.

Purpose and objectives

This problem can be solved if, while seeking function extremum, the concept of symmetry is used. It has shown itself to good advantage in such one-dimensional optimization methods as dichotomy methods, Fibonacci and the golden section. In these methods, two representative points are moving to the function extremum, symmetrically substantially reducing uncertainty (localization) interval. Similar situation is observed while determining the roots of nonlinear one-dimensional function using a combined chords-tangents method.

Main part

It is possible to carry out the improvement of multivariate methods of function extremum seeking based on the application of the concept of symmetry in the following way [3].

First, we write the expression of a downward-convex function f (x) (x - vector argument), whose extremum is found in the form:

$$f(x) = 0,5\left((x-x)^T Q(x-x) + f(x) + f(x)\right) \quad (3)$$

where Q is a positive definite symmetric matrix.

Then, replacing in (3) one of the vectors x by vector y, and the second vector x – by vector z, we obtain the auxiliary function:

$$F(y,z) = 0,5\left[(y-z)^T Q(y-z) + f(y) + f(z)\right]. \quad (4)$$

It is obvious that the extremum of function (4) will be the case when $y = z = x^*$, where x^* is the value of a vector argument for which the function $f(x)$ takes the extreme value.

The movement to the minimum of auxiliary function $F(y,z)$ is provided by the simultaneous coherent change of vector arguments y and z by any known extremum seeking algorithm.

The algorithm of the steepest gradient descent method when working with $F(y,z)$ has the following form:

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$$\frac{dy}{dt} = -kQ\left(y - z\right) - 0,5k\ grad\, f\left(y\right), \quad y\left(0\right) = y_0$$

$$y_0 \neq z_0. \qquad (5)$$

$$\frac{dz}{dt} = -kQ\left(z - y\right) - 0,5k\ grad\, f\left(z\right), \quad z\left(0\right) = z_0,$$

Due to the symmetrical algorithm **(5)** the process of convergence of variables y and z is described by the vector differential equation:

$$\frac{d\left(y - z\right)}{dt} = -2kQ\left(y - z\right) - 0,5k\left(grad\, f\left(y\right) - grad\, f\left(z\right)\right).$$

In contrast to the differential equation **(2)**, which can be represented as a system of two first order differential equations and which describes the movement of one representative point, the algorithm **(5)** describes energy interaction of two representative points. These points form a uniform system.

It is quite common in everyday life. For example, two persons by acting cohesively overcome a high hurdle: first, one of them pushes up the other, and then the second one pulls the first. Such examples are numerous.

In the case when some of the converging coordinate points are common, auxiliary function can be written as follows:

$$F\left(y_1, z_1, x_2\right) = 0,5\left[\left(y_1 - z_1\right)^T Q_1\left(y_1 - z_1\right) + f\left(y_1, x_2\right) + f\left(z_1, x_2\right)\right], \qquad (6)$$

where $f\left(y_1, x_2\right)$ and $f\left(z_1, x_2\right)$ are functions obtained from a function $f\left(x\right)$ whose minimum is searched by replacing a component x_1 of the vector $x = \left[x_1, x_2\right]$ by vectors y_1 and z_1, respectively; Q_1 is a positive definite symmetric matrix.

The algorithm **(2)** of the heavy ball method **[4]** is more effective with the auxiliary function $F(y,z)$:

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + \frac{\partial F\left(y,z\right)}{\partial y} = 0,$$

$$m\frac{d^2z}{dt^2} + r\frac{dz}{dt} + \frac{\partial F\left(y,z\right)}{\partial z} = 0. \qquad (7)$$

It should be noted that for the extremum seeking of the auxiliary function **(4)** there can be involved both continuous and discrete algorithms; several converging points. The latter allows them to overcome local extremes.

Let's consider how the principle of the symmetry concept works by applying it to the function:

$$f(x) = k \cdot (x-a)^2 - c \cdot \cos(2\pi x) + b, \qquad (8)$$

the graph shown in Fig.1.a. From Fig.1.a it is clear that the function under consideration has local extremes, which are located close to the global extremum located at the point with the coordinate x=4.

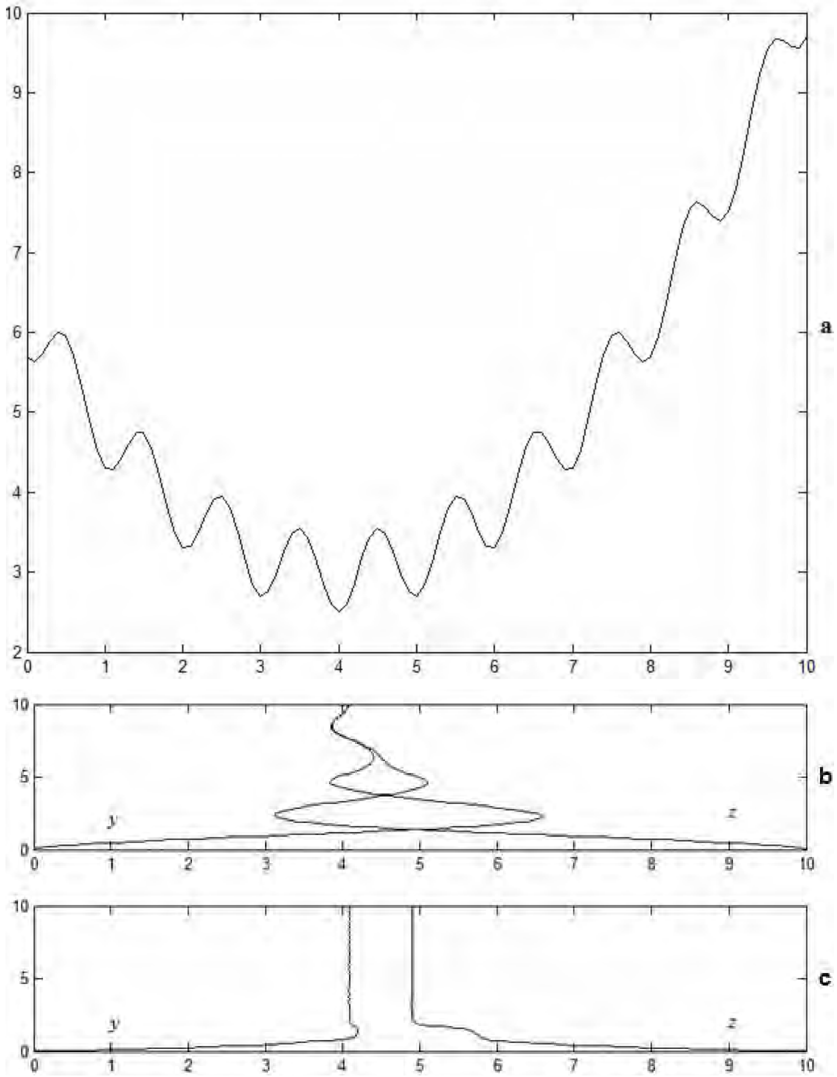

Fig. 1. a - graph of the function (8), c=0.5; the movement process of representative points to the global extremum according to the algorithm of the heavy ball method - b, the gradient method - c.

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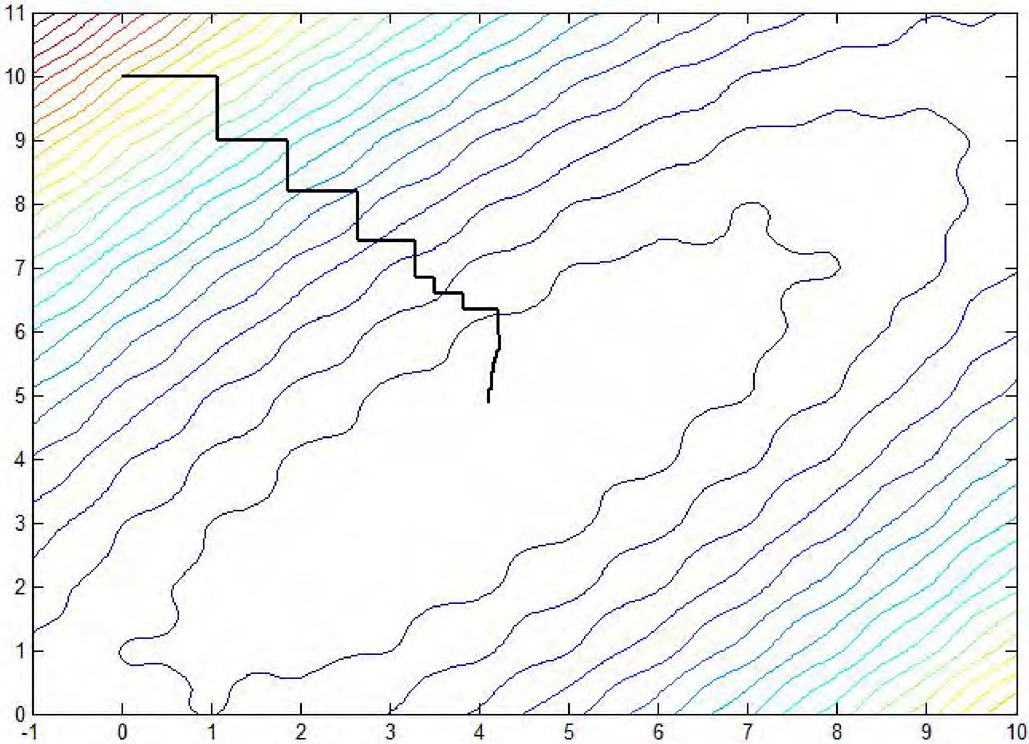

Fig. 2. Graphic illustration of movement to a minimum by the method of coordinate descent, c=0.5

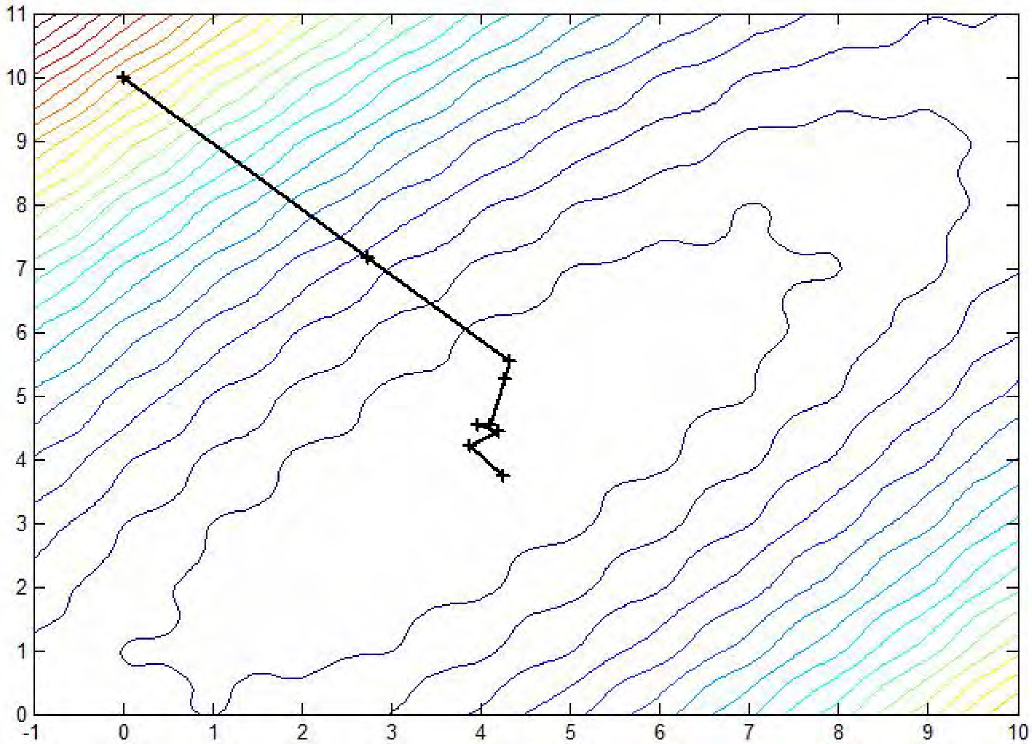

Fig. 3. Graphic illustration of movement to a minimum by the gradient descent method with constant pitch, c=0.5

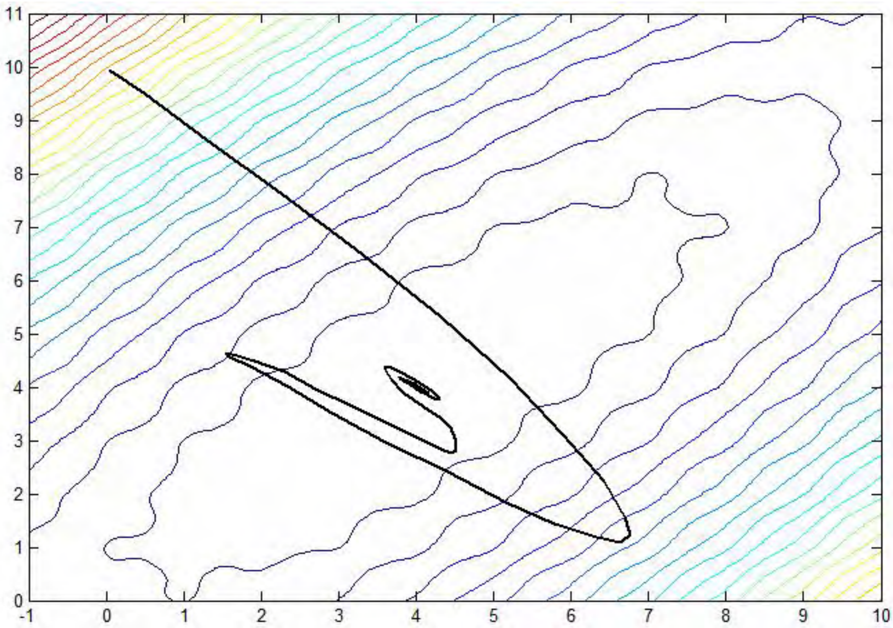

Fig. 4. Graphic illustration of movement to a minimum by the heavy ball method, c=0.5

Using equation (4), for the function (8), we obtain the auxiliary function $F(y,z)$:

$$F\left(y,z\right)=0,5\left[k\left(y-a\right)^2-c\cdot\cos(2\pi y)+k\left(z-a\right)^2-c\cdot\cos(2\pi z)+2b\right]+0.5q\left(y-z\right)^2. \quad (9)$$

The movement of two representative points to the minimum function $f\left(x\right)$ according to the algorithm of the heavy ball method is shown in Fig.1.b, gradient method – in Fig.1.c., the initial values of coordinates being y=0 and z=10 and belonging to the original function (8).

To minimize the function (9), we apply the following methods: coordinate descent, gradient method with constant pitch, and heavy ball method. The contour lines of the function (9) with the movement trajectory of representative points to its minimum, according to algorithms of the above methods (when $k=0.2,\ a=4,\ b=3,\ c=0.5,$ and $q=1$) are presented in Fig.2-4.

As it is seen from the graph (Fig.2) the method of coordinate descent does not give positive results when seeking the global extremum of the function (8), one of the representative points got trapped in a local extremum. The method of steepest descent with a constant pitch (fig.3), at the chosen step 0.254, solves this problem, but with very low accuracy (calculated values are: y=4.2529and z=3.7272).

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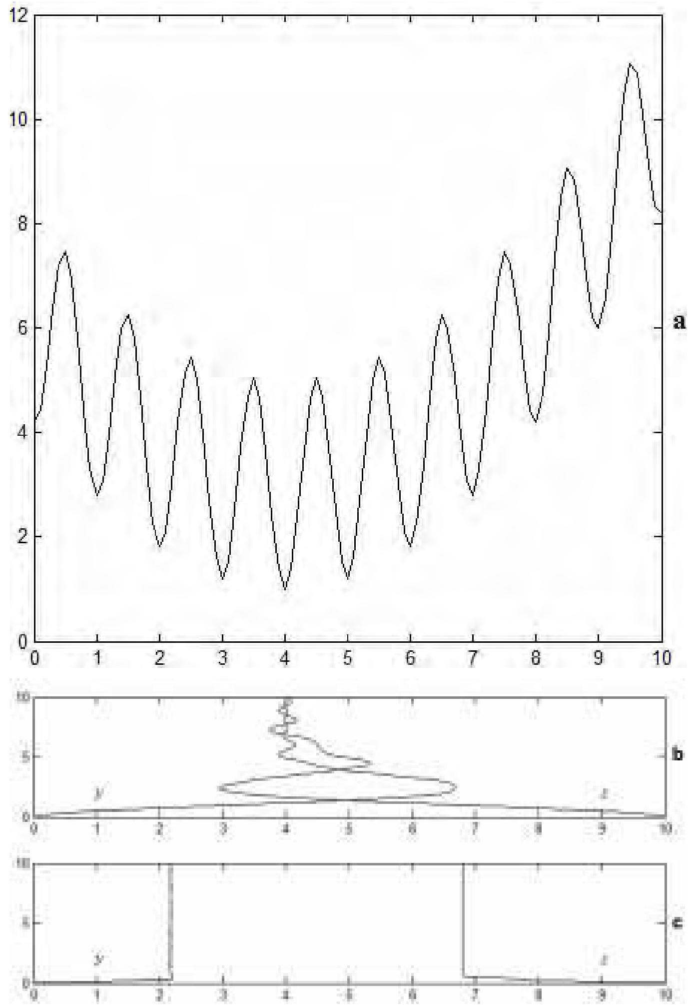

Fig. 5. a - graph of the function (8), c=2; the process of movement of representative points to the global extremum according to the algorithm of the heavy ball method - b, the gradient method - c.

Despite the multiextremal character of given function, the use of heavy ball method, for variation of the variables y and z, allows to get from an initial point (0; 10) to the finite, where $y=z=x^{*}=4$ (Fig. 4).This result was obtained with the following chosen values: the damping coefficient r = 2, the weight of the heavy balls m = 2.16. The calculated values y=3.9975 and z=4.0017 indicate that the coordinate of the global extremum of function (8) is found quite accurately.

Let's find out how the above algorithms work in finding the global extremum of the function (8), taking the parameter *c=2*, which will increase the oscillation amplitude. Graph of the function (8) with c=2 is shown in Fig.5.a.Fig.5.b illustrates the process of movement of representative points to the global extremum according to the algorithm of the heavy ball method, Fig.5.c - according to the algorithm of the gradient method.

The results of algorithms of coordinate descent method, gradient descent method with constant pitch and the hard ball method with *c=2* are shown in Fig.6-8 respectively.

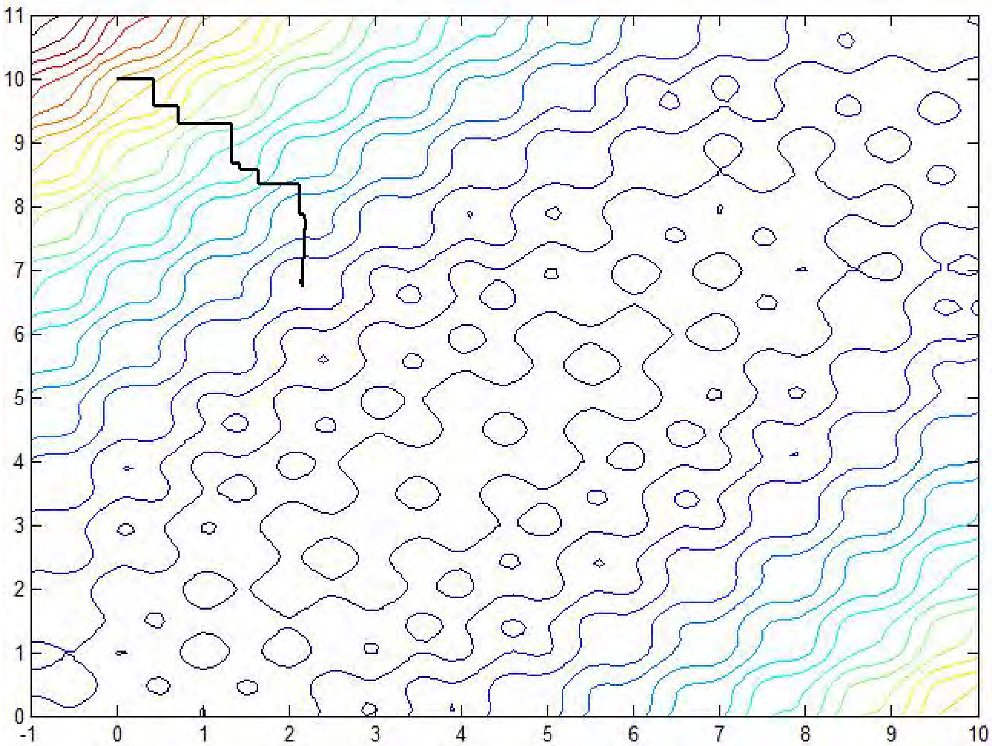

Fig. 6. Graphic illustration of movement to a minimum by the method of coordinate descent, c=2

Under increased amplitude in the methods of coordinate descent and gradient descent with a constant pitch, representative points end their movement in a local extremums. In the heavy ball method, the movement process of representative points ends at the global optimum $x^*=4$. The calculated values were: $y=4.0006$ and $z=3.9973$, at the weight of heavy balls $m=2.3$ and the damping coefficient $r=2$.

The results of the above algorithms for c = 0.5 and c = 2 are summarized in Table 1.

Table 1

Method	*c*=0.5		*c*=2	
	the number of iterations	*x**	the number of iterations	*x**
coordinate descent	20	$\begin{pmatrix}4.0840\\4.8919\end{pmatrix}$	63	$\begin{pmatrix}2.1491\\6.8420\end{pmatrix}$
gradient method with constant pitch	9	$\begin{pmatrix}4.2529\\3.7272\end{pmatrix}$	8	$\begin{pmatrix}4.6124\\4.1595\end{pmatrix}$
hard ball	127	$\begin{pmatrix}3.9975\\4.0017\end{pmatrix}$	144	$\begin{pmatrix}3.9973\\4.0006\end{pmatrix}$

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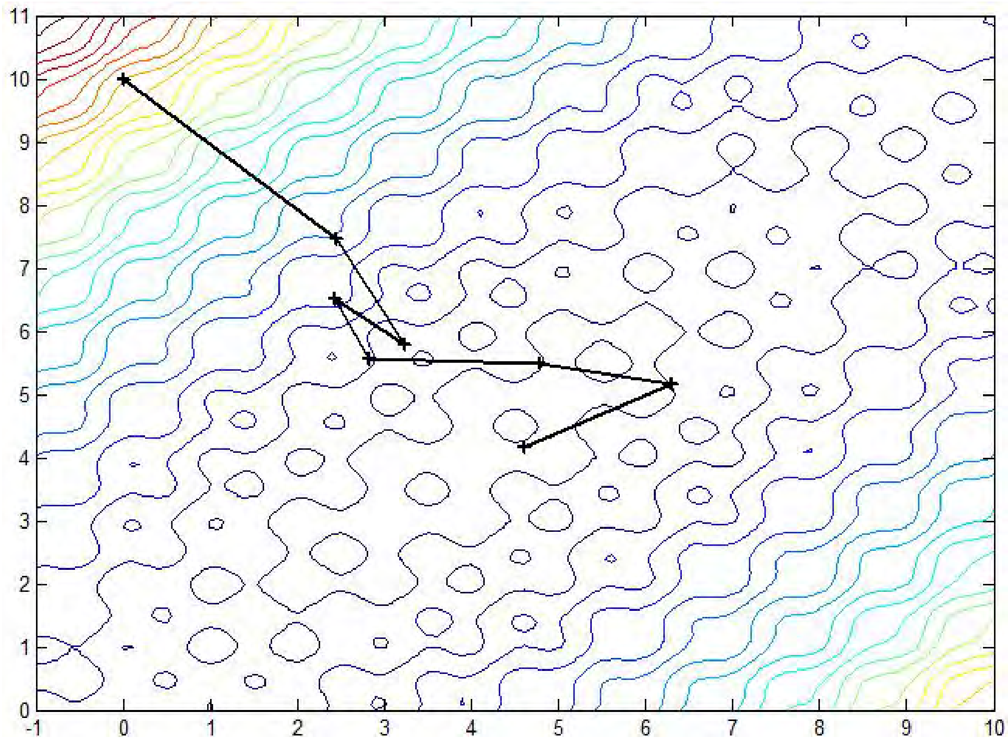

Fig. 7. Graphic illustration of movement to a minimum by the gradient descent method with constant pitch, c=2

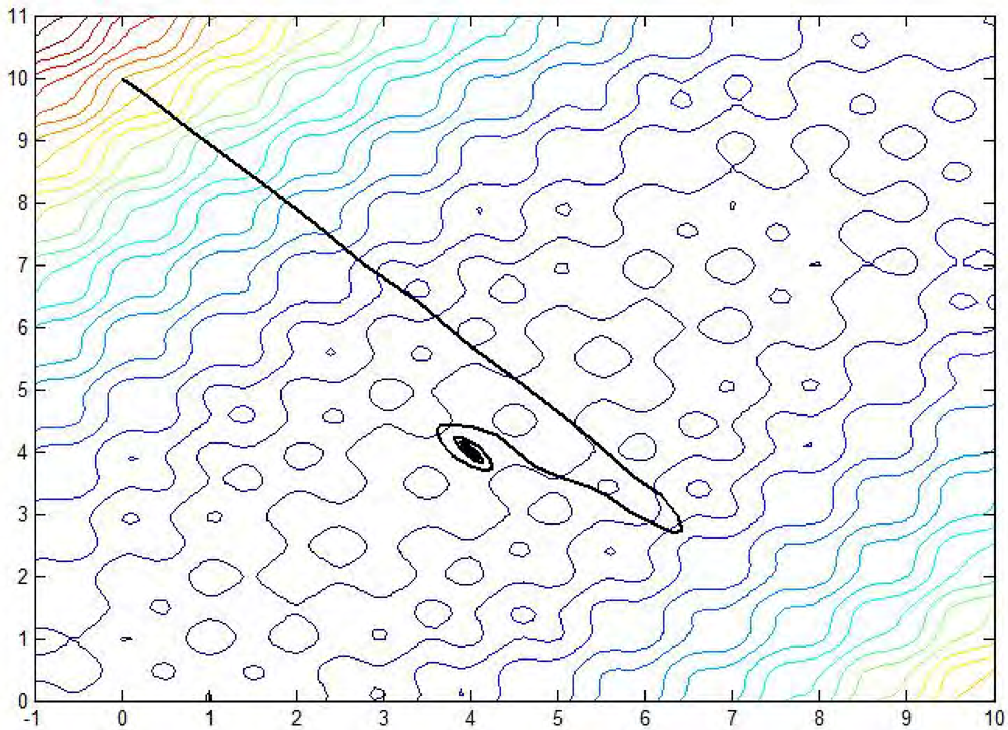

Fig.8. Graphic illustration of movement to a minimum by the heavy ball method, c=2

Conclusions and perspectives for further research

Analyzing the obtained results, we can draw the following conclusions:

1. Gradient descent method with a constant pitch, as compared with the heavy ball method, with $c=0.5$ finds the global extremum of the function at the minimal number of iterations, but has low accuracy.
2. With an increase in the oscillation amplitude to $c=2$ gradient methods are of no use for seeking the global extremum of function as the representative points finish their movement in local minimums.
3. Application the principle of symmetry to the algorithm of heavy ball method solves the task of finding the global extremum of the function (8) with high accuracy, even at increased amplitude of oscillation.
4. Parallelization of extremum seeking of function based on the use of the concept of symmetry applied to a class of dynamic optimization problems has allowed [5,6] and in the future will allow obtaining a number of positive results for the estimation of unknown parameters of objects.

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