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ON THE EXISTENCE OF H^1-SOLUTIONS TO CERTAIN IMAGE REGISTRATION PROBLEMS

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The solubility of the class of nonlinear optimization problems arising in image registration is discussed. The necessary optimality conditions (Euler-Lagrange equation) for such kind of problems is a nonlinear Neumann boundary value problem which is not known to have a solution in general. However, in the image registration context some assumptions can be made that let us move a little bit further in this question.

Key words. Image registration, nonlinear optimization problem, nonlinear Neumann boundary value problem, existence of H^1-solutions.

1. Introduction

Consider the following optimization problem:

$$J(u) = \int_\Omega |I^T(x - u(x)) - I^R(x)|^2\, dx + a(u, u) \to \inf, \tag{1.1}$$

where $u(x)$ is an *offset field* associated with the transformation $\varphi(x) = x - u(x)$, u is an element of a certain class of admissible displacements W; and the application dependent regularizing term $a(\cdot, \cdot)$ is a bilinear bounded form in $W \times W$. The *images* I^T (template) and I^R (reference) are nonnegative functions in Ω. The domain Ω in R^d where images are defined is assumed to be bounded and Lipschitz. We also assume that I^T can be extended by zero to R^d so that $I^T(x - u(x))$ makes sense for any u.

This is a so-called image registration problem for the monomodal images, i.e. obtained on the same hardware so that the intensity of their pixels can be compared directly, as in the first term in J which is a sum of squared differences (SSD). An example of reference and template images is in Figure 1 which shows two sets of orthogonal multiplanar projections of a human femur taken withtin the interval of one year. The purpose of *monomodal* registration in this case is to compare changes in different subvolumes of interest caused by a medical treatment.

From this point on, the reference image $I^R(x)$ is supposed to be an element of the space $L^\infty(\Omega)$. The template image $I^T(x)$ is an element of $C(\overline{\Omega}) \cap C^1(\Omega)$.

The regularizing term $a(\cdot, \cdot)$, a bilinear form, penalizes the undesired properties of u.

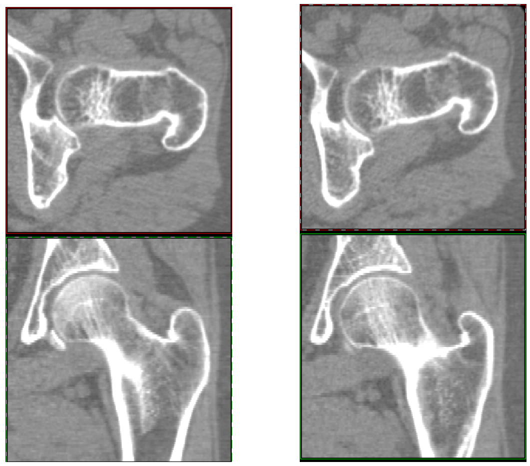

Fig. 1. Example of a template and a reference image in R^3: Orthogonal multiplanar projections of a CT-scan of human femur. Left and right show slices from two volumes taken withtin the time interval of one year, extracted at approximately the same anatomical position.

The existence of solutions to the problem (1.1) for several conventional regularizers $a(\cdot,\cdot)$ (diffusion, elastic, etc) can be readily stated in Sobolev space $W = H_0^1(\Omega)^d$ since the functional $J(u)$ is lower semicontinuous and coercive in the weak topology of this space (see [3], where also classical solutions are considered). See also [2], where the well-posedness of the Euler-Lagrange system for (1.1) is considered in abstract Banach space (of course, the solubility of the system doesn't guarantee the existence of minimizers for original problem).

However, when computing numerical solutions to an image registration problem, one usually works in the space $W = H^1(\Omega)^d$, see [1] for instance. The reason is that the Euler-Lagrange system for the problem of type (1.1), used to compute a solution, is usually supplied with *homogenous Neumann* boundary conditions on u. These conditions are natural in image registration. Indeed, looking for the suitable registration transformation $\varphi(x) = x - u(x)$ which makes a template being similar to a reference (in the sense of our image similarity measure (1.1)), we would like to admit a non-zero offset for every pixel in a moving template image: consider the rigid rotation as a typical example in image matching. Such a possibility would be lost with the homogenous Dirichlet boundary conditions, which are applied in a case of the space $H_0^1(\Omega)^d$. To the best knowledge of the author, the existence of H^1-solutions to the problem (1.1) was not yet established.

Summarizing, we can say that in the image registration area the problem (1.1) is de facto considered in $H^1(\Omega)^d$, which is, despite of the great amount of important practical results, remains formal problem. The purpose of the paper is to fill up this gap between the theoretical and practical parts of the image registration problem.

2. Assumptions and the proof of existence

According to the direct methods in the calculus of variations, the weak lower semicontinuity and coerciveness of the functional are sufficient to state the existence of the minimizers. We are going to show the weak lower semicontinuity of the functional $J(u)$ in $H^1(\Omega)$ and the weak sequential compactness of the minimizing sequence u_n only, not the coercivity of the functional J. The argument of the direct methods is then equally applicable.

The check of the lower semicontinuity is almost straightforward, which is not the case with the compactness. To obtain the compactness we apply the idea of the proof of the Poincaré inequality for the sequence of functions u_n which are known to be bounded in $H^1(D)$ where D is a subset of Ω.

First of all we establish the following auxiliary result.

Proposition 1. Let a sequence $\{U_n\}$ of subsets in Ω be such that there exist a sequence of points $\{x_n\}$ in Ω which, together with the balls of some radius $\varepsilon > 0$ around them, belong to the corresponding sets U_n: $B_\varepsilon(x_n) \subset U_n$ for all n.

Then there exist a point y and a ball $B_\delta(y)$ in Ω which belong to every set in a subsequence of the sets U_n:

$$B_\delta(y) \subset U_{n_k}, \qquad k = 1, 2 \dots$$

for some $\delta > 0$.

Proof. Let us fix a number $\delta < \varepsilon$ and consider a mesh with grid points $y_i \in \Omega$, $i \in \overline{1, l}$ such that the distance between any two points in a single cell is less then $\varepsilon - \delta$. Assume that the corollary claimed in Proposition is false. For the point y_1 it means that there exists an index m such that $B_\delta(y_1) \not\subset U_k$ for all $k > m$. That is, $|x_k - y_1| > \varepsilon - \delta$. Now take the rest points $y_2, y_3, ...y_l \in \Omega$ in turn. Since the domain Ω is bounded, only a finite number of balls $B_\varepsilon(x_k)$ can belong to Ω. This contradiction concludes the proof. $\square$

The following lemma is used to prove the boundedness of the minimizing sequence u_n in the space $H^1(D)$ where D is a certain subset of Ω.

Lemma 1. *Let Ω be bounded. Let for a sequence $\{u_n \in L^2(\Omega)^d\}$ there exists $\varepsilon > 0$ and a sequence of balls $B_\varepsilon(x_n)$ of radius ε in Ω such that*

$$\sup_n \|u_n\|_{L^2(B_\varepsilon(x_n))^d} < \infty. \tag{2.1}$$

Then there exist a subsequence u_{n_k} and such a set $D \subset \Omega$ that its Lebesgue volume is positive $\mathfrak{L}^d(D) > 0$ and

$$\sup_k \|u_{n_k}\|_{L^2(D)^d} < \infty.$$

Proof. From Proposition 1 we readily get the existence of a ball $B_\delta(y)$ in Ω such that $B_\delta(y) \subset B_\varepsilon(x_{n_k})$ for a certain subsequence of the set sequence $B_\varepsilon(x_n)$. That is,

$$\sup_k \|u_{n_k}\|_{L^2(B_\delta(y))^d} < \infty \qquad \text{for all } n,$$

which concludes the proof with $D = B_\delta(y)$. $\square$

To get use of Lemma 1 and ensure the boundedness of gradients of the minimizing sequence, we suppose appropriate properties of the registration problem (1.1).

(A) There exist $\varepsilon > 0$, $\delta > 0$, $C > 0$ and a sequence of balls $B_\varepsilon(x_n)$ of radius ε in Ω such that if

$$\|\overline{u}\|_{L^2(B_\varepsilon(x_n))^d} > C \text{ for all } B_\varepsilon(x_n)$$

then

$$\int\limits_\Omega |I^T(x - \overline{u}(x)) - I^R(x)|^2\,dx \geq \inf_{u \in H^1(\Omega)^d} \int\limits_\Omega |I^T(x - u(x)) - I^R(x)|^2\,dx + \delta.$$

(B) The continuous bilinear form $a(\cdot,\cdot)$ is coercive in H_0^1-norm.

Remark 1. In fact, it is natural even to assume that $\|\overline{u}\|_{L^\infty(B_\varepsilon(x))^d} > C$ in assumption (A) above, since otherwise we admit the existence of a mapping $\overline{u}$ which is (sub)optimal (the value of the functional on it can be arbitrarily close to the infimum) and such that any neighbourhood in Ω contains another neighbourhood which is sent outside of Ω by $\overline{u}$.

The condition (B) is fulfilled in the above mentioned cases (elastic, diffusion registration, etc).

We are now in a position to establish the main result.

Theorem 1. *Given the problem* (1.1) *let the conditions* (A)–(B) *are satisfied.*

Then there is a function $u^0 \in H^1(\Omega)^d$ such that $J(u^0) \leq J(u)$ for all $u \in H^1(\Omega)^d$.

Proof. Let $u_n \in H^1(\Omega)^d$ be a minimizing sequence for the problem (1.1).

Compactness of the minimizing sequence. Since the bilinear form $a(u,u)$ is coercive in H_0^1-norm (condition (B)), the sequences $\{\nabla u_n^i\}$ are bounded in $L^2(\Omega)^d$, $i \in \overline{1,d}$. Thus, we need to establish the boundedness of $\{u_n\}$ in $L^2(\Omega)^d$ to ensure the compactness of the minimizing sequence in $H^1(\Omega)^d$. Using condition (A), we have from Lemma 1 the boundedness of $\{u_n\}$ in $L^2(D)^d$ for some $D \subset \Omega$ (up to a subsequence). Then the boundedness in $L^2(\Omega)^d$ directly follows from the Poincaré inequality.

Consequently, one can find a subsequence of $\{u_n\}$ (not relabelled), which converges weakly in $H^1(\Omega)^d$ to some u^0.

Weak lower semicontinuity. Since u_n is weakly convergt in $H^1(\Omega)^d$, it is also strongly convergent in $L^2(\Omega)^d$ and, therefore, is convergent in measure. Then, by the virtue of Fatou's Lemma and continuity of $a(\cdot,\cdot)$ we obtain

$$\begin{aligned}\int\limits_\Omega &\left|I^T(x - u^0) - I^R(x)\right|^2 dx + a(u^0,u^0)\\ &\leq \liminf_{n\to\infty} \int\limits_\Omega \left|I^T(x - u_n) - I^R(x)\right|^2 dx + a(u_n,u_n).\end{aligned}$$

That is, u^0 is a minimizer for $J(u)$. $\square$

3. Closing remarks

The existence of solutions for a certain class of image registration problems is established. To this end, a strong assumption on the behaviour of minimizing sequence is done which however relates to the specifics of the registration problem.

The result can be extended to other registration problems which are based not on SSD-term but on the mutual information (MI) or the cross-correlation (CC), for example. On the other hand, the proposed argument is also applicable for the registration problem with certain other regularizers when formulated in an appropriate space. One example is a *curvature* registration problem in $H^2(\Omega)^d$ with $a(u,u) = \int\limits_{\Omega} |\Delta u|^2 \, dx$.

The condition (A), although being quite natural, is hard to verify. Therefore, the future efforts will be to find an alternative assumption which is based on the properties of images only and doesn't use variational properties of the registration problem.

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