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H-OPTIMAL CONTROL IN COEFFICIENTS FOR DIRICHLET PARABOLIC PROBLEMS

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In this paper we study the Dirichlet optimal control problem associated with a linear parabolic equation the coefficients of which we take as controls in $L^1(\Omega)$. Since equations of this type can exhibit the Lavrentieff phenomenon and non-uniqueness of weak solutions, we show that the optimal control problem in the coefficients can be stated in different forms depending on the choice of the class of admissible solutions. Using the direct method in the Calculus of variations, we discuss the solvability of the above optimal control problems in the so-called class of H-admissible solutions.

Key words. Degenerate parabolic equations, control in coefficients, weighted Sobolev spaces, Lavrentieff phenomenon, direct method in the Calculus of variations.

1. Introduction

The aim of this work is to study the optimal control problems associated to a linear parabolic equation with homogeneous Dirichlet boundary condition. The control variable is the matrix of L^1-coefficients in the main part of elliptic operator. The precise answer existence or none-existence of an L^1-optimal solutions heavily depends on the class of admissible controls. The main questions are what is the right setting of the optimal control problem with L^1-controls in the coefficients, and what is the right class of admissible solutions to the above problem? Using the direct method in the Calculus of variations, we discuss the solvability of the above optimal control problems in the class of H-admissible solutions.

Note that optimal control problems in coefficients for PDE are not new in the literature. As François Murat showed in 1970 (see [14]), in general, such problems have no solution even if the original elliptic equation is non-degenerate. It turns out that this feature is typical for the majority of problems for optimal control in coefficients. Note that this topic has been widely studied by many authors in the case of non-degenerate weight function. In this paper we deal with an optimal control problem in coefficients for the boundary value problem

$$\begin{cases} y' - \operatorname{div} B(x)\nabla y + y = f & \text{in } \ (0,T)\times\Omega, \\ y = 0 & \text{on } \ (0,T)\times\partial\Omega, \\ y(0,x) = y_0(x) & \text{a. e. in } \ \Omega, \end{cases} \tag{1.1}$$

where $f \in L^2((0,T) \times \Omega)$ and $y_0 \in L^2(\Omega)$ are given functions, and B is a non negative invertible matrix such that $B + B^{-1} \in L^1(\Omega; \mathbb{R}^{N\times N})$. Several physical phenomena are modeled by this parabolic problem. In order to be able to handle media which possibly are "perfect" insulators somewhat or "perfect" conductors (see [8]) we allow the matrix B to vanish somewhere in Ω or to be unbounded.

Even though numerous papers (see, for instance, [6, 16, 17, 21] and references there) are devoted to variational and non variational approaches to problems related to (1.1), only few papers deal with optimal control problems for degenerate partial differential equations (see, for example, [1, 3, 5]). This can be explained by several reasons. Firstly, boundary value problem (1.1) for every locally integrable matrix B exhibit the Lavrentieff phenomenon, the non-uniqueness of weak solutions, as well as other surprising consequences. So, in general, the mapping $B \mapsto y(B)$ can be multi-valued. Besides, the characteristic feature of this problem is the fact that for different admissible controls B with properties prescribed above, the corresponding weak solutions of (1.1) belong to the different weighted Sobolev spaces. In addition, even if the original parabolic equation is non-degenerate, i.e. admissible controls B are such that

$$B(x) \geq \alpha I, \quad (B(x))^{-1} \geq \beta^{-1} I, \quad \text{a.e. in } \Omega,$$

the majority of optimal control problems in coefficients have no solution.

Our paper is organized as follow: at the beginning we state problem of optimal control in the coefficients and prescribe the class of admissible controls which includes some div-like conditions in weighted spaces. After that we discuss the classification of admissible solutions to the above optimal control problem. We show that one of the characteristic features of this problem is the following fact: for every admissible L^1-control the corresponding H-solution to the boundary value problem belongs to a weighted space which essentially depends on the original control. So, the set of the so-called H-admissible solutions to the above problem can be viewed as a collection of pairs "control-state"in the variable spaces each of which is embedded into $L^1(\Omega; \mathbb{R}^{N\times N}) \times L^2(0,T; W_0^{1,1}(\Omega))$.

Further we deal with the existence of optimal solutions to the original problem. We begin with a refinement of the celebrated div-curl lemma of F. Murat and L.C. Tartar [15] to the case of variable weighted Sobolev spaces. After we study the topological properties of the class of H-admissible solutions and show that this set possesses some compactness properties with respect to the appropriate convergence in variable spaces. In conclusion, using the direct method in the Calculus of variations, we prove the existence of the H-optimal solutions to the original problem.

2. Notation and Preliminaries

In this section we introduce some notation and preliminaries that will be useful later on.

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$ with a Lipschitz boundary. Let χ_E be the characteristic function of a subset $E \subseteq \Omega$, i.e. $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$. The space $W_0^{1,1}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the classical Sobolev space $W^{1,1}(\Omega)$. For any subset $E \subset \Omega$ we denote by $|E|$ its

N-dimensional Lebesgue measure $\mathcal{L}^N(E)$. Let $M_\alpha^\beta(\Omega)$ be the set of all matrices $A = [a_{ij}]$ in $L^\infty(\Omega; \mathbb{R}^{N\times N})$ such that

$$A(x) \geq \alpha I, \quad (A(x))^{-1} \geq \beta^{-1} I, \quad \text{a.e. in } \Omega \tag{2.1}$$

for two fixed constants α and β with $0 < \alpha \leq \beta < +\infty$. Here I is the identity matrix in $\mathbb{R}^{N\times N}$, and inequalities (2.1) should be considered in the sense of the quadratic forms defined by $(A\xi, \xi)_{\mathbb{R}^N}$ for $\xi \in \mathbb{R}^N$. Note that (2.1) implies the inequality $|A(x)| \leq \beta$ a.e. in Ω.

Hereinafter by a weight we mean a locally integrable function ρ on $\mathbb{R}^N$ such that $\rho(x) > 0$ for a. e. $x \in \mathbb{R}^N$. As a matter of fact every weight ρ gives rise to a measure on the measurable subsets of $\mathbb{R}^N$ through integration. This measure will also be denoted by ρ. Thus $\rho(E) = \int_E \rho\, dx$ for measurable sets $E \subset \mathbb{R}^N$. We will use the standard notation $L^2(\Omega, \rho\, dx)$ for the set of measurable functions f on Ω such that

$$\|f\|_{L^2(\Omega,\rho\, dx)} = \left(\int_\Omega f^2 \rho\, dx\right)^{1/2} < +\infty.$$

Definition 1. We say that a weight function $\rho : \mathbb{R}^N \to \mathbb{R}_+$ is degenerate on Ω if

$$\rho + \rho^{-1} \in L^1_{loc}(\mathbb{R}^N), \tag{2.2}$$

and the sum $\rho + \rho^{-1}$ does not belong to $L^\infty(\Omega)$.

With each of the degenerate weight functions ρ we will associate two weighted Sobolev spaces $W_\rho = W(\Omega, \rho\, dx)$ and $H_\rho = H(\Omega, \rho\, dx)$, where W_ρ is the set of functions $y \in W_0^{1,1}(\Omega)$ for which the norm

$$\|y\|_\rho = \left(\int_\Omega \left(y^2 + \rho\, |\nabla y|^2\right)\, dx\right)^{1/2} \tag{2.3}$$

is finite, and H_ρ is the closure of $C_0^\infty(\Omega)$ in W_ρ-norm. Note that due to the compact embedding $W_0^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ and estimates

$$\int_\Omega |y|\, dx \leq |\Omega|^{1/2} \left(\int_\Omega |y|^2\, dx\right)^{1/2} \leq \sqrt{|\Omega|}\, \|y\|_\rho, \tag{2.4}$$

$$\int_\Omega |\nabla y|\, dx \leq \left(\int_\Omega |\nabla y|^2 \rho\, dx\right)^{1/2} \left(\int_\Omega \rho^{-1}\, dx\right)^{1/2} \leq C\|y\|_\rho, \tag{2.5}$$

we come to the following result (we refer to [11, 21] for the details):

Theorem 1. *Let $\rho : \mathbb{R}^N \to \mathbb{R}_+$ be a degenerate weight on Ω. Then*

(i) the spaces H_ρ and W_ρ are complete with respect to the norm $\|\cdot\|_\rho$;

(ii) $H_\rho \subseteq W_\rho$, and W_ρ, H_ρ are Hilbert spaces;

(iii) $H_\rho \subset W_0^{1,1}(\Omega)$, $W_\rho \subset W_0^{1,1}(\Omega)$, and the estimate

$$\|v\|_{W_0^{1,1}(\Omega)} \leq \left(\sqrt{|\Omega|} + \left(\int_\Omega \rho^{-1}\, dx\right)^{1/2}\right) \|v\|_\rho$$

is valid for every element $v \in H_\rho \cup W_\rho$;

(iv) the embeddings $H_\rho \hookrightarrow L^1(\Omega)$ *and* $W_\rho \hookrightarrow L^1(\Omega)$ *are compact.*

If ρ is a non-degenerate weight function, that is, ρ is bounded between two positive constants, then it is easy to verify that $W_\rho = H_\rho$. However, for a "typical" degenerate weight ρ the space of smooth functions $C_0^\infty(\Omega)$ is not dense in W_ρ. Hence the identity $W_\rho = H_\rho$ is not always valid (for the corresponding examples we refer to [7, 19].

We recall that by Riesz Representation Theorem the dual space $(H_\rho)^*$ of weighted Sobolev space H_ρ can be characterized as follows: if $g \in (H_\rho)^*$ then there exist functions $g_0 \in L^2(\Omega)$ and $\overrightarrow{g}_1 \in L^2(\Omega, \rho\, dx)^N$ such that

$$\langle g, y\rangle_{(H_\rho)^*;H_\rho} = \int_\Omega g_0 y\, dx + \int_\Omega (\overrightarrow{g}_1, \nabla y)_{\mathbb{R}^N}\, \rho\, dx \quad \forall\, y \in H_\rho. \tag{2.6}$$

Furthermore,

$$\|g\|_{(H_\rho)^*} = \inf\left\{ \left(\int_\Omega |g_0|^2\, dx + \int_\Omega \|\overrightarrow{g}_1\|^2_{\mathbb{R}^N} \rho\, dx \right)^{1/2} \ :\ g \ \text{ satisfies (2.6)} \right\}.$$

We denote by H_ρ^{-1} the dual space to H_ρ.

Remark 1. Note that under some additional suppositions Theorem 1 can be specified as follows: assume that there exists $\nu \in (N/2, +\infty)$ such that $\rho^{-\nu} \in L^1(\Omega)$. Then

$$|||y||| = \left(\int_\Omega \rho\, |\nabla y|^2\, dx \right)^2$$

is a norm defined on H_ρ and it's equivalent to (2.3) and that, the embedding $H_\rho \hookrightarrow L^2(\Omega)$ is compact [9, pp 46].

To conclude this section we recall some results concerning variational triplets. Let $V_- = H_\rho$, $V = L^2(\Omega)$ and let $V_-^* = H_\rho^{-1}$. Let $\mathcal{X} = L^2(0,T;V_-)$. Then the dual space of $\mathcal{X}$ is $\mathcal{X}^* = L^2(0,T;V_-^*)$. For any $y \in \mathcal{X}$, let y' denotes the generalized derivative of $y(t) = y(t,\cdot)$, i.e.

$$\int_0^T y'(t)\varphi(t)\, dt = -\int_0^T y(t)\varphi'(t)\, dt \quad \forall\, \varphi \in C_0^\infty(0,T).$$

Then we have the following result (see [18]):

Lemma 1. *Assume that there exists* $\nu \in (N/2, +\infty)$ *such that* $\rho^{-\nu} \in L^1(\Omega)$. *Then* $V_- \subseteq V \subseteq V_-^*$ *is an evolution triple, i.e. the embeddings* $V_- \hookrightarrow V \hookrightarrow V_+$ *are continuous, and the embedding* $V_- \hookrightarrow V$ *is compact. Moreover,* $\mathcal{W} = \{y \in \mathcal{X},\ y' \in \mathcal{X}^*\}$ *equipped with the norm*

$$\|y\|_{\mathcal{W}} = \|y\|_{\mathcal{X}} + \|y'\|_{\mathcal{X}^*} := \|y\|_{L^2(0,T;H_\rho)} + \|y'\|_{L^2(0,T;H_\rho^{-1})}$$

is a Banach space such that

1. *the embedding* $\mathcal{W} \hookrightarrow C(0,T;L^2(\Omega))$ *is continuous;*
2. *the embedding* $\mathcal{W} \hookrightarrow L^2(0,T;L^2(\Omega))$ *is compact.*

3. Setting of the Optimal Control Problem

Let ρ be given element of $L^1(\Omega)$ satisfying the conditions

$$0 < \rho(x) \text{ a.e. in } \Omega, \quad \rho^{-\nu} \in L^1(\Omega) \text{ for some } \nu \in (N/2, +\infty). \tag{3.1}$$

Then, in view of the estimate

$$\int_\Omega \rho^{-1}\, dx \le \left(\int_\Omega \rho^{-\nu}\, dx\right)^{1/\nu} \left(\int_\Omega dx\right)^{1/\nu^*} = \|\rho^{-\nu}\|^{1/\nu}_{L^1(\Omega)} |\Omega|^{1/\nu^*},$$

where $\nu^* = \nu/(1-\nu)$ is the conjugate of ν, we have: $\rho^{-1} \in L^1(\Omega)$, i.e., ρ is a degenerate weight in the sense of Definition 1. In order to introduce the class of admissible L^1-controls, we adopt the following concept:

Definition 2. For a given $\vec{v} \in \left[L^2(\Omega, \rho\, dx)\right]^N$ we say that an element $g \in L^2(\Omega, \rho\, dx)$ is the divergence of the vector field $\vec{v}$ with respect to the weight ρ (in symbols $g(x) = \operatorname{div}_\rho \vec{v}(x)$) if $\vec{v}$ and g are related by the formula

$$\int_\Omega g(x)\varphi(x)\, \rho(x)\, dx = -\int_\Omega (\vec{v}(x), \nabla\varphi(x))_{\mathbb{R}^N}\ \rho(x)\, dx \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{3.2}$$

Definition 3. We say that a matrix $B \in L^1(\Omega; \mathbb{R}^{N\times N})$ is an admissible control (it is written as $B \in \mathcal{B}_{ad}$) to the parabolic problem

$$y' - \operatorname{div} B(x)\nabla y + y = f \quad \text{in} \quad (0,T)\times\Omega, \tag{3.3}$$
$$y(0,x) = y_0 \quad \text{a.e. in} \quad \Omega, \tag{3.4}$$
$$y = 0 \quad \text{on} \quad (0,T)\times\partial\Omega \tag{3.5}$$

if there is a symmetric matrix $A = [\vec{a}_1, \dots, \vec{a}_N] \in L^\infty(\Omega; \mathbb{R}^{N\times N})$ such that

$$B(x) = A(x)\rho(x), \quad A \in M_\alpha^\beta(\Omega), \tag{3.6}$$
$$|\operatorname{div}_\rho \vec{a}_i| \le \gamma_i \quad \rho - \text{a. e. in} \quad \Omega, \quad \forall\, i = 1, \dots, N, \tag{3.7}$$

where $f \in L^2(\Omega)$, $y_0 \in L^2(\Omega)$, $\gamma = (\gamma_1, \dots, \gamma_N) \in \mathbb{R}^N$ is a given positive vector, elements $\operatorname{div}_\rho \vec{a}_i \in L^2(\Omega, \rho\, dx)$ are defined by (3.2). Here ρ is the fixed element of $L^1(\Omega)$ with properties (3.1).

Remark 2. As follows from Definition 3 and properties (3.1), for every admissible control $B \in L^1(\Omega; \mathbb{R}^{N\times N})$ we deal with the initial-boundary value problem for the degenerate parabolic equation

$$y' - \operatorname{div}(\rho A(x)\nabla y) + y = f \quad \text{in} \quad (0,T)\times\Omega, \tag{3.8}$$
$$y(0,x) = y_0 \ \text{ a.e. in } \ \Omega, \quad y = 0 \ \text{ on } \ (0,T)\times\partial\Omega. \tag{3.9}$$

It means that for some admissible matrices of coefficients $B \in \mathcal{B}_{ad}$ the boundary value problem (3.3)–(3.5) can exhibit the Lavrentieff phenomenon [19] as well as other surprising consequences.

The optimal control problem we consider in this paper is to minimize the discrepancy between a given distribution $y_d \in L^2((0,T)\times\Omega)$ and the solution of the parabolic problem (3.3)–(3.5) by choosing an appropriate matrix of coefficients $B \in \mathcal{B}_{ad}$. More precisely, we are concerned with the following optimal control problem

$$\text{Minimize} \quad \Big\{ I(B,y) = \zeta \int_0^T \int_\Omega |y(t,x) - y_d(t,x)|^2 \, dxdt \\ + \int_0^T \int_\Omega |\nabla y(x)|^2_{\mathbb{R}^N} \rho \, dxdt + \|A\|_{L^\infty(\Omega, R^{N\times N})} \Big\} \text{ subject constraints (3.6)–(3.7).} \tag{3.10}$$

Here $\zeta > 0$ is a penalization parameter.

Let $B = A\rho \in \mathcal{B}_{ad}$ be an admissible control. Then the quadratic form

$$\Phi(y) = \int_\Omega A(x)\nabla y \cdot \nabla y \, \rho \, dx$$

with domain $W_\rho \subset L^2(\Omega)$ is closed and corresponds to a non-negative self-adjoint operator $\mathcal{A}_W = -\mathrm{div}\, \rho A \nabla$ in $L^2(\Omega)$. At the same time this form will also be closed in $H_\rho \subset L^2(\Omega)$, which leads us to another non-negative self-adjoint operator $\mathcal{A}_H = -\mathrm{div}\, \rho A \nabla$ in $L^2(\Omega)$. Thus, there exist at least two different problems

$$y' + \mathcal{A}_W y + y = f \quad \text{and} \quad y' + \mathcal{A}_H y + y = f, \tag{3.11}$$

relating to initial-boundary value problem (3.3)–(3.5). As we will see later, each of the problem (3.11) is uniquely solvable. So, the mapping $B \mapsto y(B,f)$, where $y(B,f)$ is a solution to problem (3.3)–(3.5), is multivalued, in general.

4. Classification of optimal solutions

In view of the observation given above, we adopt the classification of the solutions to the initial-boundary valued problem (3.3)–(3.5) following Pastukhova & Zhikov [21] (for more details and other types of solutions we refer to [2, 11, 20]).

Definition 4. We say that a function $y = y(B,f) = y(A,\rho,f) \in L^2(0,T;W_\rho)$ is a weak solution to the initial-boundary value problem (3.3)–(3.5) for a fixed control $B = A\rho \in \mathcal{B}_{ad}$ and a given function $f \in L^2((0,T)\times\Omega)$, if for each $\varphi \in C_0^\infty(\Omega)$, y satisfies the integral identity

$$\int_\Omega y\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega \Big(\Big(A(x)\, \nabla y, \nabla \varphi \Big)_{\mathbb{R}^N} \rho(x) + y\varphi \Big) \, dxdt \\ = \int_{t_1}^{t_2} \int_\Omega f\varphi \, dxdt \quad \forall\, t_1, t_2 \in [0,T], \tag{4.1}$$

and

$$\lim_{t\to+0} \int_\Omega y\varphi \, dx = \int_\Omega y_0 \varphi \, dx \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{4.2}$$

As immediately follows from Definition 4 that a weak solution $y(t,\cdot)$ is weakly continuous as a function $[0,T] \to W_\rho$ and $y|_{t=0} = y_0$. This follows from the boundedness of $y(t,\cdot) : [0,T] \to W_\rho$ and the continuity of the functions

$$\int_\Omega y(t,x)\varphi(x)\,dx \quad \text{on} \quad [0,T].$$

Definition 5. Let V_ρ be some intermediate space with $H_\rho \subseteq V_\rho \subseteq W_\rho$. We say that a function $y = y(A,\rho,f) \in L^2(0,T;V_\rho)$ is a V_ρ-solution or a variational solution to the initial-boundary value problem (3.3)–(3.5) if y satisfies condition (4.2) and the integral identity (4.1) for every test function $\varphi \in V_\rho$.

Remark 3. Note that for every fixed $B = A\rho \in \mathcal{B}_{ad}$ the existence and uniqueness of a V_ρ-solution can be established using the standard technique [13]. Moreover, if $V_\rho = H_\rho$, them, in view of Lemma 1, we have: H_ρ-solution to (3.3)–(3.5) possesses the additional properties

$$y \in \mathcal{W} = \left\{ y \in L^2(0,T;H_\rho),\ y' \in L^2(0,T;H_\rho^{-1}) \right\},$$

and hence $y \in C(0,T;L^2(\Omega))$. At the same time, the variational solutions do not exhaust the entire set of the weak solutions to the above boundary value problem. Indeed, by analogy with [21] it can be proved that a weak solution $y = y(B,f) \in L^2(0,T;W_\rho)$ is a variational one if and only if, in addition to (4.1)–(4.2), the energy equality

$$\frac{1}{2}\int_\Omega y^2\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega \left(\Big(A(x)\,\nabla y, \nabla y\Big)_{\mathbb{R}^N}\rho + y^2 \right) dxdt = \int_{t_1}^{t_2}\int_\Omega f y\,dxdt \quad (4.3)$$

holds true for all $t_1, t_2 \in [0,T]$. Therefore, if $y_1(B,f), y_2(B,f) \in W_\rho$ are variational solutions with $y_1(B,f) \neq y_2(B,f)$ (hence they belong to the different intermediate spaces $V_{1,\rho}$ and $V_{2,\rho}$), then $y = (y_1(B,f) + y_2(B,f))/2$ is a weak solution to (3.3)–(3.5) but not variational one. Moreover, as follows from Definition 4 the set of weak solutions to the initial-boundary value problem (3.3)–(3.5) for a fixed control $B = A\rho \in \mathcal{B}_{ad}$ is convex and closed. Hence if $y_1(B,f), y_2(B,f)$ are variational solutions such that $y_1(B,f) \neq y_2(B,f)$ then the corresponding set of the weak solutions is infinite.

It is obvious that for every fixed $B \in \mathcal{B}_{ad}$, $f \in L^2((0,T)\times\Omega)$, and $V_\rho(H_\rho \subseteq V_\rho \subseteq W_\rho)$ a variational solution is also a weak solution to the problem (3.3)–(3.5). However, the inverse assertion is not true in general. For a "typical" degenerate weight function ρ the space of smooth functions $C_0^\infty(\Omega)$ is not dense in W_ρ, and hence there is no uniqueness of the weak solutions (see, for instance, [12, 20]).

Now it is clear that the mapping $B \mapsto y(B,f)$ can be viewed as multi-valued in general, and this depends on the choice of the corresponding solutions space V_ρ. As a result, the variational formulation of the optimal control problem (3.6)–(3.7),(3.10) can be stated in different forms. Taking this fact into account, we indicate the following sets

$$\Xi_H = \{(B,y)\ |\ B = A\rho \in \mathcal{B}_{ad},\ y \in H_\rho, (B,y) \text{ are related by } (4.1)\text{–}(4.3)\}, \quad (4.4)$$

$$\Xi_W = \{(B,y)\ |\ B = A\rho \in \mathcal{B}_{ad},\ y \in W_\rho, (B,y) \text{ are related by } (4.1)\text{–}(4.3)\}. \quad (4.5)$$

As was mentioned above (see Remark 3), the sets Ξ_H and Ξ_W are always non-empty. Hence the corresponding minimization problems

$$\left\langle \inf_{(B,y)\in\,\Xi_H} I(B,y)\right\rangle \quad \text{and} \quad \left\langle \inf_{(B,y)\in\,\Xi_W} I(B,y)\right\rangle \tag{4.6}$$

are regular. However, because of the Lavrentieff effect, it may happen that for some fixed control $B = A\rho \in \mathcal{B}_{ad}$ and a given $f \in L^2((0,T)\times\Omega)$ the corresponding H_ρ-solution $y_H(A,\rho,f)$ and W_ρ-solution $y_W(A,\rho,f)$ to the initial-boundary value problem (3.8)–(3.9) are not the same. This implies that the variational problems (4.6) are essentially different, in general. Hence, the minimizers to (4.6) can be also different, and moreover

$$\inf_{(B,y)\in\,\Xi_H} I(B,y) \neq \inf_{(B,y)\in\,\Xi_W} I(B,y).$$

Note that due to the Remark 3 and estimates (2.4)–(2.5), we have the obvious inclusions

$$\begin{aligned}\Xi_H \subset L^1(\Omega;\mathbb{R}^{N\times N}) \times L^2(0,T;H(\Omega,\rho\,dx)) \cap C(0,T;L^2(\Omega)),\\ \Xi_W \subset L^1(\Omega;\mathbb{R}^{N\times N}) \times L^2(0,T;W(\Omega,\rho\,dx)).\end{aligned}$$

In this paper we restrict of our analysis to the set Ξ_H and adopt the following concept:

Definition 6. We say that a pair

$$(B^0,y^0) \in L^1(\Omega;\mathbb{R}^{N\times N}) \times L^2(0,T;H_\rho) \cap C(0,T;L^2(\Omega))$$

is an H-optimal solution to the problem (3.3)–(3.7),(3.10) if

$$(B^0,y^0) \in \Xi_H \quad \text{and} \quad I(B^0,y^0) = \inf_{(B,y)\in\,\Xi_H} I(B,y).$$

The main question for the optimal control problem (3.3)–(3.7),(3.10) to be answered in this paper is about its solvability in the class of H-solutions. It should be noted that to the best knowledge of the authors, the existence of optimal pairs to the above problem in the sense of Definition 6 has not been studied in the literature.

5. On Compensated Compactness in Weighted Sobolev Spaces

We begin this section with some auxiliary results that will be useful later. Let $\{(B_k,y_k) = (A_k\rho, y_k) \in \Xi_H\}_{k\in\mathbb{N}}$ be any sequence of H-admissible solutions. With this sequence we associate the space

$$X_\rho = \left\{\vec{f} \in L^2(0,T;L^2(\Omega,\rho\,dx))^N \mid \operatorname{div}_\rho \vec{f} \in L^2(0,T;L^2(\Omega,\rho\,dx))\right\}$$

and endow it with the norm

$$\|\vec{f}\|_{X_\rho} = \left(\|\vec{f}\|^2_{L^2(0,T;L^2(\Omega,\rho\,dx))^N} + \|\operatorname{div}_\rho \vec{f}\|^2_{L^2(0,T;L^2(\Omega,\rho\,dx))}\right)^{1/2}.$$

We say that a sequence $\left\{\vec{f}_k \in X_\rho\right\}_{k\in\mathbb{N}}$ is bounded if

$$\sup_{k\to\infty} \|\vec{f}\|_{X_\rho} < +\infty.$$

Further, for every $k > 0$ we define a cut-off operator $T_k : \mathbb{R} \to \mathbb{R}$ as follows $T_k(s) = \max\{\min\{s, k\}, -k\}$. By analogy with the well-known results for the classical Sobolev spaces (see [10]), it is easy to verify the following assertion:

Proposition 1. Let y be an element of the weighted space

$$\mathcal{W} = \left\{y \in L^2(0,T; H_\rho),\ y' \in L^2(0,T; H_\rho^{-1})\right\}.$$

Then

(i) $T_k(y) \in \mathcal{W}$ for every $k > 0$;

(ii) $\nabla_x T_k(y) = \chi_{\{|y|<k\}} \nabla_x y$ almost everywhere in Ω;

(iii) $T_k(y) \to y$ almost everywhere in $(0,T) \times \Omega$ and strongly in $L^2(0,T; H_\rho)$ as $k \to \infty$.

Taking these properties and Proposition 2.3 from [11] into account, by the diagonal trick, we come to the conclusion:

Proposition 2. Let ρ be an element of $L^1(\Omega)$ with properties (3.1). Let $\{g_k \in \mathcal{W}\}_{k\in\mathbb{N}}$ be a bounded sequence such that

$$\begin{array}{rll} g_k \rightharpoonup g & \text{in} & L^2((0,T);\Omega), \\ \nabla g_k \rightharpoonup \nabla g & \text{in} & L^2(0,T; L^2(\Omega, \rho\,dx))^N, \qquad \text{as} \ \ k \to \infty. \\ g_k' \rightharpoonup g' & \text{in} & L^2(0,T; H_\rho^{-1}) \end{array} \tag{5.1}$$

Then there exists an increasing sequence of positive numbers $\{\ell_k\}_{k\in\mathbb{N}}$ such that $\ell_k \to +\infty$ as $k \to \infty$, and

$$T_{\ell_k}(g_k) \to g \quad \text{strongly in} \ \ L^2(0,T; L^2(\Omega)) \ \text{as} \ \ k \to \infty. \tag{5.2}$$

Now we are in the position to give the main result of this section (for comparison we refer to the Compensated Compactness Lemma in [4, 15]).

Theorem 2. *Let* $\left\{\vec{f}_k \in L^2(0,T; L^2(\Omega, \rho\,dx))^N\right\}_{k\in\mathbb{N}}$, $\vec{f} \in L^2(0,T; L^2(\Omega, \rho\,dx))^N$, $\{g_k \in \mathcal{W}\}_{k\in\mathbb{N}}$, *and* $g \in \mathcal{W}$ *be such that*

(i) $\vec{f}_k \rightharpoonup \vec{f}$ *in* $L^2(0,T; L^2(\Omega, \rho\,dx))^N$ *as* $k \to \infty$;

(ii) $g_k \rightharpoonup g$ *in* $L^2((0,T) \times \Omega)$, $\nabla g_k \rightharpoonup \nabla g$ *in* $L^2(0,T; L^2(\Omega, \rho_k\,dx))^N$, *and* $g_k' \rightharpoonup g'$ *in* $L^2(0,T; H_\rho^{-1})$ *as* $k \to \infty$.

Then

$$\lim_{k\to\infty} \int_0^T \int_\Omega \varphi \left(\vec{f}_k, \nabla g_k\right)_{\mathbb{R}^N} \rho\psi dx dt = \int_0^T \int_\Omega \varphi \left(\vec{f}, \nabla g\right)_{\mathbb{R}^N} \rho\psi\, dx dt, \tag{5.3}$$
$$\forall\, \varphi \in C_0^\infty(\Omega),\ \forall\, \psi \in C_0^\infty(0,T).$$

Proof. We divide our proof into several steps. Our first step is to prove that

$$\operatorname{div}_\rho \vec{f}_k \rightharpoonup \operatorname{div}_\rho \vec{f} \quad \text{in} \quad L^2(0,T;L^2(\Omega,\rho\,dx)) \text{ as } \; k\to\infty. \tag{5.4}$$

Indeed, since the sequence $\left\{\operatorname{div}_\rho \vec{f}_k \in L^2(0,T;L^2(\Omega,\rho\,dx))\right\}_{k\in\mathbb{N}}$ is bounded, by the compactness criterium in reflexive spaces, we can suppose that there exists an element $\phi\in L^2(0,T;L^2(\Omega,\rho\,dx))$ such that

$$\operatorname{div}_\rho \vec{f}_k \rightharpoonup \phi \quad \text{in} \quad L^2(0,T;L^2(\Omega,\rho\,dx)) \text{ as } \; k\to\infty.$$

Then passing to the limit in the relation

$$\int_0^T\int_\Omega \left(\vec{f}_k,\nabla\varphi\right)_{\mathbb{R}^N}\rho\psi\,dxdt = -\int_0^T\int_\Omega \varphi\,\left(\operatorname{div}_\rho \vec{f}_k\right)\rho\psi\,dxdt \tag{5.5}$$

$\forall\,\varphi\in C_0^\infty(\Omega)$, $\forall\,\psi\in C_0^\infty(0,T)$ as $k\to\infty$, we obtain

$$\int_0^T\int_\Omega \left(\vec{f},\nabla\varphi\right)_{\mathbb{R}^N}\rho\psi\,dxdt = -\int_0^T\int_\Omega \varphi\,\phi\rho\psi\,dxdt$$
$$\forall\,\varphi\in C_0^\infty(\Omega),\quad \forall\,\psi\in C_0^\infty(0,T).$$

Therefore (see Definition 2), the element ϕ is the anisotropic divergence of the vector field $\vec{f}\in L^2(0,T;L^2(\Omega,\rho\,dx))^N$ with respect to the weight ρ, i.e., $\phi = \operatorname{div}_\rho \vec{f}\in L^2(0,T;L^2(\Omega,\rho\,dx))$. So, (5.4) is valid.

The next step is to study the asymptotic behavior as $k\to+\infty$ of the following numerical sequence

$$\left\{\int_0^T\int_\Omega \varphi\,\left(\vec{f}_k,\nabla g_k\right)_{\mathbb{R}^N}\rho\psi\,dxdt\right\}_{k\in\mathbb{N}}.$$

To begin with, we note that as follows from Lemma 1, the element $g\in\mathcal{W}$ is the strong limit of $\{g_k\in\mathcal{W}\}_{k\in\mathbb{N}}$ in $L^2((0,T)\times\Omega)$-topology. So, we can suppose that

$$g_k\to g \quad \text{a. e. in } \; (0,T)\times\Omega. \tag{5.6}$$

In view of estimates

$$\left|\int_\Omega \left(\vec{f}_k,\nabla\varphi\right)_{\mathbb{R}^N}\rho\,dx\right| \le \left(\int_\Omega \left\|\vec{f}_k\right\|^2_{\mathbb{R}^N}\rho\,dx\right)^{1/2}\left(\int_\Omega \|\nabla\varphi\|^2_{\mathbb{R}^N}\rho\,dx\right)^{1/2},$$
$$\left|\int_\Omega \varphi\,\left(\operatorname{div}_\rho \vec{f}_k\right)\rho\,dx\right| \le \|\varphi\|_{L^\infty(\Omega)}\|\rho\|^{1/2}_{L^1(\Omega)}\left(\int_\Omega \left(\operatorname{div}_\rho \vec{f}_k\right)^2\rho\,dx\right)^{1/2}$$

and by density of $C_0^\infty(\Omega)$ in H_ρ, the relation (5.5) can be extended to the functions φ of $H_\rho\cap L^\infty(\Omega)$. Since $T_\ell(g_k)\in L^\infty(0,T;H_\rho\cap L^\infty(\Omega))$ for every $k\in\mathbb{N}$ and $\ell\in\mathbb{N}$, it follows that

$$\int_0^T\int_\Omega \left(\vec{f}_k,\nabla\left(T_\ell(g_k)\varphi\right)\right)_{\mathbb{R}^N}\rho\psi\,dxdt = -\int_0^T\int_\Omega \left(\operatorname{div}_\rho \vec{f}_k\right)\,\varphi\,T_\ell(g_k)\rho\psi\,dxdt$$

for all $\varphi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(0,T)$. Due to this relation, we make use the following equality

$$\int_0^T \int_\Omega \varphi \left(\vec{f}_k, \nabla T_\ell(g_k)\right)_{\mathbb{R}^N} \rho\psi \, dxdt = \int_0^T \int_\Omega \left(\vec{f}_k, \nabla \left(T_\ell(g_k)\varphi\right)\right)_{\mathbb{R}^N} \rho\psi \, dxdt$$
$$- \int_0^T \int_\Omega T_\ell(g_k) \left(\vec{f}_k, \nabla\varphi\right)_{\mathbb{R}^N} \rho\psi \, dxdt = - \int_0^T \int_\Omega \left(\operatorname{div}_\rho \vec{f}_k\right) \varphi \, T_\ell(g_k)\rho\psi \, dxdt$$
$$- \int_0^T \int_\Omega T_\ell(g_k) \left(\vec{f}_k, \nabla\varphi\right)_{\mathbb{R}^N} \rho\psi \, dxdt$$
$$= -I_{1,\ell}^k - I_{2,\ell}^k \quad \forall \, \varphi \in C_0^\infty(\Omega), \ \forall \, \psi \in C_0^\infty(0,T). \quad (5.7)$$

Our next intention is to study the asymptotic behavior of the integrals $I_{1,\ell}^k$ and $I_{2,\ell}^k$ as $k \to \infty$. Since the sequence $\left\{\operatorname{div}_\rho \vec{f}_k \in L^2(\Omega, \rho \, dx)\right\}_{k\in\mathbb{N}}$ is bounded, the property (5.4) implies that

$$\rho \operatorname{div}_\rho \vec{f}_k \rightharpoonup \rho \operatorname{div}_\rho \vec{f} \quad \text{in} \quad L^1((0,T)\times\Omega). \quad (5.8)$$

Hence the family $\left\{\rho \operatorname{div}_\rho \vec{f}_k\right\}_{k\in\mathbb{N}}$ is equi-integrable on $(0,T)\times\Omega$. Therefore, because of the boundedness of $\{T_\ell(g_k) - T_\ell(g)\}$ the sequence

$$\left\{\rho \, \left(T_\ell(g_k) - T_\ell(g)\right) \operatorname{div}_\rho \vec{f}_k\right\}_{k\in\mathbb{N}} \quad \text{is equi-integrable on} \quad (0,T)\times\Omega$$

as well. Using the property (5.6), we have

$$T_\ell(g_k) \to T_\ell(g) \quad \text{a. e. in} \quad (0,T)\times\Omega \quad \text{for every} \quad \ell \in \mathbb{N}.$$

Then Lebesgue's Theorem implies

$$\rho \, \left(T_\ell(g_k) - T_\ell(g)\right) \operatorname{div}_\rho \vec{f}_k \to 0 \quad \text{in} \quad L^1((0,T)\times\Omega) \quad \text{as} \quad k\to\infty.$$

Moreover, by (5.8), we get

$$T_\ell(g)\rho \operatorname{div}_\rho \vec{f}_k \rightharpoonup T_\ell(g)\rho \operatorname{div}_\rho \vec{f} \quad \text{in} \quad L^1((0,T)\times\Omega) \quad \text{as} \quad k\to\infty.$$

Combining these results, we obtain

$$\rho T_\ell(g_k) \operatorname{div}_\rho \vec{f}_k = \rho \left(T_\ell(g_k) - T_\ell(g)\right) \operatorname{div}_\rho \vec{f}_k$$
$$+ \rho T_\ell(g) \operatorname{div}_\rho \vec{f}_k \rightharpoonup \rho T_\ell(g) \operatorname{div}_\rho \vec{f} \ \text{in} \ L^1((0,T)\times\Omega). \quad (5.9)$$

On the other hand, the inequality

$$\left\|T_\ell(g_k) \operatorname{div}_\rho \vec{f}_k\right\|_{L^2(0,T;L^2(\Omega,\rho\, dx))} \le \|T_\ell(g_k)\|_{L^\infty((0,T)\times\Omega)}$$
$$\times \left\|\operatorname{div}_\rho \vec{f}_k\right\|_{L^2(0,T;L^2(\Omega,\rho\, dx))} \le C,$$

immediately yields that $\left\{T_\ell(g_k) \operatorname{div}_\rho \vec{f}_k\right\}_{k\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(\Omega,\rho dx))$ for every $\ell\in\mathbb{N}$. Hence, there exists an element $\eta^\ell \in L^2(0,T;L^2(\Omega,\rho\, dx))$ such that

$$T_\ell(g_k) \operatorname{div}_\rho \vec{f}_k \rightharpoonup \eta^\ell \ \text{in} \ L^2(0,T;L^2(\Omega,\rho dx)),$$

that is, $T_\ell(g_k)\rho\,\mathrm{div}_\rho\,\vec{f}_k \rightharpoonup \eta^\ell\,\rho$ in $L^1((0,T)\times\Omega)$. Then, in view of (5.9), we get

$$\eta^\ell = T_\ell(g)\,\mathrm{div}_\rho\,\vec{f} \quad \rho\text{-almost everywhere in } \ \Omega.$$

As a result, we come to the relation

$$\lim_{k\to\infty} I^k_{1,\ell} = \int_0^T \int_\Omega T_\ell(g)\,\varphi\,\mathrm{div}_\rho\,\vec{f}\,\rho\psi\,dxdt. \tag{5.10}$$

Using similar arguments, we can prove that

$$\lim_{k\to\infty} I^k_{2,\ell} = \int_0^T \int_\Omega T_\ell(g)\,\left(\vec{f},\nabla\varphi\right)_{\mathbb{R}^N}\rho\psi\,dxdt.$$

Thus, the passage to the limit in (5.7) leads us to the relation

$$\begin{aligned}
&\lim_{k\to\infty}\int_0^T \int_\Omega \varphi\,\left(\vec{f}_k,\nabla T_\ell(g_k)\right)_{\mathbb{R}^N}\rho\psi\,dxdt\\
&\quad= -\int_0^T \int_\Omega T_\ell(g)\varphi\,\mathrm{div}_\rho\,\vec{f}\,\rho\psi\,dxdt - \int_0^T \int_\Omega T_\ell(g)\,\left(\vec{f},\nabla\varphi\right)_{\mathbb{R}^N}\rho\psi\,dxdt\\
&\quad= \int_0^T \int_\Omega \left(\vec{f},\nabla\left(T_\ell(g)\varphi\right)\right)_{\mathbb{R}^N}\rho\psi\,dxdt - \int_0^T \int_\Omega T_\ell(g)\,\left(\vec{f},\nabla\varphi\right)_{\mathbb{R}^N}\rho\psi\,dxdt\\
&\quad= \int_0^T \int_\Omega \varphi\,\left(\vec{f},\nabla T_\ell(g)\right)_{\mathbb{R}^N}\rho\psi\,dxdt \quad \forall\,\varphi\in C_0^\infty(\Omega),\ \forall\psi\in C_0^\infty(0,T)
\end{aligned}\tag{5.11}$$

which holds true for every $\ell\in\mathbb{N}$.

Let $\{T_{\ell_k}(g_k)\in H_\rho\}_{k\in\mathbb{N}}$ be a sequence with properties (i)–(iii) which is ensured by Proposition 1. Then for any $\delta>0$ there exists a value $k^*\in\mathbb{N}$ such that

$$\left(\int_0^T \|T_{\ell_k}(g_k)-g_k\|^2_\rho\,dt\right)^{1/2} \le \delta \quad \forall\,k>k^* \quad \text{(by Proposition 1)}.$$

By Cauchy-Bunyakovskiǐ inequality we have the estimate

$$\begin{aligned}
L &= \sup_{k\in\mathbb{N}}\left|\int_0^T \int_\Omega \varphi\,\left(\vec{f}_k,\nabla T_{\ell_k}(g_k)-g_k\right)_{\mathbb{R}^N}\rho\psi\,dxdt\right|\\
&\le \delta\|\varphi\|_{C(\overline{\Omega})}\|\psi\|_{C(0,T)}\|\vec{f}_k\|_{L^2(0,T;L^2(\Omega,\rho\,dx))^N} \le C\delta.
\end{aligned}\tag{5.12}$$

Taking into account that $\chi_{\{|g_k|<\ell_k\}}\to\chi_\Omega$ strongly in $L^\infty((0,T)\times\Omega)$, it finally follows that

$$\begin{aligned}
&\left|\lim_{k\to\infty}\int_0^T \int_\Omega \varphi\,\left(\vec{f}_k,\nabla g_k\right)_{\mathbb{R}^N}\rho\psi\,dxdt - \int_0^T \int_\Omega \varphi\,\left(\vec{f},\nabla g\right)_{\mathbb{R}^N}\rho\psi\,dxdt\right|\\
&\quad\overset{\text{by (5.12)}}{\le} \left|\lim_{k\to\infty}\int_0^T \int_\Omega \varphi\,\left(\vec{f}_k,\nabla T_{\ell_k}(g_k)\right)_{\mathbb{R}^N}\rho\psi\,dxdt\right.\\
&\qquad\left. - \int_0^T \int_\Omega \varphi\,\left(\vec{f},\nabla g\right)_{\mathbb{R}^N}\rho\psi\,dxdt\right| + C\delta\\
&\quad\overset{\text{by (5.11)}}{\le} \left|\lim_{k\to\infty}\int_0^T \int_\Omega \chi_{\{|g_k|<\ell_k\}}\varphi\,\left(\vec{f},\nabla g\right)_{\mathbb{R}^N}\rho\psi\,dxdt\right.\\
&\qquad\left. - \int_0^T \int_\Omega \varphi\,\left(\vec{f},\nabla g\right)_{\mathbb{R}^N}\rho\psi\,dxdt\right| + C\delta \overset{\text{(by Proposition 1)}}{=} C\delta.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, this concludes the proof. $\square$

Remark 4. The key point of the proof of this lemma is the fact that the space of smooth functions $C_0^\infty(\Omega)$ is dense in the weighted space $H_\rho = H(\Omega, \rho dx)$. So, in general, Lemma 2 does not hold for the case when $\{g_k\}_{k\in\mathbb{N}}$ is a bounded sequence in the weighted Sobolev space W_ρ.

6. Existence Theorem for H-optimal solutions

Our prime interest in this section deals with the solvability of optimal control problem (3.3)–(3.7),(3.10) in the class of H-solutions. To begin with, we consider the topological properties of the set of H-admissible solutions Ξ_H to the problem (3.3)–(3.7),(3.10). To do so, we introduce the following concepts:

Definition 7. We say that a sequence $\{(B_k, y_k) = (A_k\rho_k, y_k) \in \Xi_H\}_{k\in\mathbb{N}}$ is bounded if

$$\sup_{k\in\mathbb{N}} \Big[\|A_k\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} + \|y_k'\|_{L^2(0,T;H_\rho^{-1})} + \|y_k\|_{L^2((0,T)\times\Omega)} + \|\nabla y_k\|_{L^2(0,T;L^2(\Omega,\rho\, dx))^N}\Big] < +\infty.$$

Definition 8. We say that a bounded sequence of H-admissible solutions

$$\{(B_k, y_k) = (A_k\rho_k, y_k) \in \Xi_H\}_{k\in\mathbb{N}}$$

τ-converges to a pair $(B, y) \in L^1(\Omega;\mathbb{R}^{N\times N}) \times \mathcal{W}$ if

(a) $B = A\rho$, where $A \in L^\infty(\Omega;\mathbb{R}^{N\times N})$;

(b) $A_k \overset{*}{\rightharpoonup} A$ in $L^\infty(\Omega;\mathbb{R}^{N\times N})$;

(c) $y_k \rightharpoonup y$ in $L^2((0,T)\times\Omega)$;

(d) $\nabla y_k \rightharpoonup \nabla y$ in $L^2(0,T;L^2(\Omega,\rho dx))^N$;

(e) $y_k' \rightharpoonup y'$ in $L^2(0,T;H_\rho^{-1})$.

Theorem 3. *For every $f \in C_0^\infty(\mathbb{R}^N)$ the set Ξ_H is closed with respect to the τ-convergence.*

Proof. Let $\{(B_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi_H$ be a bounded τ-convergent sequence of H-admissible pairs to the optimal control problem (3.6)–(3.7),(3.10). Let

$$(B_0, y_0) = (A_0\rho, y_0)$$

be its τ-limit. Our aim is to prove that $(B_0, y_0) \in \Xi_H$.

In view of the initial assumptions (3.6)–(3.7) we have:

$$A_k = [\vec{a}_{1\,k}, \ldots, \vec{a}_{N\,k}] \in M_\alpha^\beta(\Omega)$$

$$\text{and} \quad |\mathrm{div}_\rho\, \vec{a}_{i\,k}| \le \gamma_i \quad \rho\, dx\text{-a.e. in } \Omega \ \ \forall\, i = 1,\ldots,N,\ \forall\, k\in\mathbb{N}.$$

Hence, the sequences $\left\{\operatorname{div}_\rho \vec{a}_{i\,k} \in L^2(\Omega,\rho\,dx)\right\}_{k\in\mathbb{N}}$ $\forall\, i = 1,\dots,N$ are uniformly bounded. The compactness criterium in $L^2(\Omega,\rho\,dx)$-spaces implies the existence of elements $\left\{\phi_i \in L^2(\Omega,\rho\,dx)\right\}_{i=1}^N$ such that

$$\operatorname{div}_\rho \vec{a}_{i\,k} \rightharpoonup \phi_i \quad \text{in} \quad L^2(\Omega,\rho\,dx) \text{ as } \ k\to\infty \quad \forall\, i=1,\dots,N.$$

Then passing to the limit as $k\to\infty$ in the relations

$$\int_\Omega (\vec{a}_{i\,k},\nabla\varphi)_{\mathbb{R}^N}\rho\,dx = -\int_\Omega \varphi \operatorname{div}_\rho \vec{a}_{i\,k}\,\rho\,dx$$
$$\forall\,\varphi\in C_0^\infty(\Omega),\ \forall\, i\in\{1,\dots,N\},\ \forall\, k\in\mathbb{N},$$
$$-\gamma_i\int_\Omega \varphi\rho\,dx \le \int_\Omega \varphi \operatorname{div}_\rho \vec{a}_{i\,k}\,\rho\,dx \le \gamma_i \int_\Omega \varphi\rho\,dx$$
$$\forall\, i\in\{1,\dots,N\},\ \forall\, k\in\mathbb{N},\ \forall\varphi\ge 0,$$
$$A_k = [\vec{a}_{1\,k},\dots,\vec{a}_{N\,k}] \in M_\alpha^\beta(\Omega),$$

we come to the conclusion:

$$\operatorname{div}_\rho \vec{a}_{i\,k} \rightharpoonup \phi_i = \operatorname{div}_\rho \vec{a}_{i\,0} \quad \text{in} \quad L^2(\Omega,\rho\,dx) \ \text{ as } \ k\to\infty, \tag{6.1}$$
$$|\operatorname{div}_\rho \vec{a}_{i\,0}| \le \gamma_i \quad \rho-\text{a.e. in } \ \Omega\ \forall\, i\in\{1,\dots,N\}, \tag{6.2}$$
$$A_k \overset{*}{\rightharpoonup} A_0 = [\vec{a}_{1\,0},\dots,\vec{a}_{N\,0}] \in M_\alpha^\beta(\Omega). \tag{6.3}$$

Hence the limit matrix $B_0 = A_0\rho$ is an admissible control to the problem (3.6)–(3.7),(3.10).

It remains to show that the pair (B_0, y_0) is related by the energy equality (4.3). We will do it in several steps. Step 1. To begin with, we note that, by the initial assumptions there exists of a constant $C>0$ such that

$$\|y_k\|_{L^2((0,T)\times\Omega)} \le C, \quad \|\nabla y_k\|_{L^2(0,T;L^2(\Omega,\rho\,dx))^N} \le C,$$
$$\|y_k'\|_{L^2(0,T;H_\rho^{-1})} \le C \quad \forall\, k\in\mathbb{N}.$$

Using the standard arguments, we can suppose that there exists an element $y_*\in\mathcal{W}$ such that, up to a subsequence we have (see also Lemma 1)

$$y_k \rightharpoonup y_* \ \text{ weakly in the Sobolev space } \ L^2(0,T;H_\rho), \tag{6.4}$$
$$y_k' \rightharpoonup y_*' \ \text{ in } \ L^2(0,T;H_\rho^{-1}), \tag{6.5}$$
$$\text{and } \ y_k \to y_* \ \text{ in } \ L^2((0,T)\times\Omega). \tag{6.6}$$

Further, we note that the sequence

$$\{A_k\nabla y_k\}_{k\in\mathbb{N}} \ \text{ is bounded in } \ L^2(0,T;L^2(\Omega,\rho\,dx))^N.$$

Hence passing to a subsequence if necessary, we may assume that there exists a function $\vec{\eta}\in L^2(0,T;L^2(\Omega,\rho\,dx))^N$ such that

$$A_k\,\nabla y_k =: \vec{\eta}_k \rightharpoonup \vec{\eta} \quad \text{in} \quad L^2(0,T;L^2(\Omega,\rho\,dx))^N. \tag{6.7}$$

Taking these facts into account, we can pass to the limit in the integral identity

$$-\int_0^T\int_\Omega y_k\varphi\psi'\,dxdt+\int_0^T\int_\Omega\Big(\Big(A_k\,\nabla y_k,\nabla\varphi\Big)_{\mathbb{R}^N}\rho+\,y_k\varphi\Big)\,\psi\,dxdt$$
$$=\int_0^T\int_\Omega f\varphi\psi\,dxdt\quad\forall\,\varphi\in C_0^\infty(\Omega),\forall\,\psi\in C_0^\infty(0,T)\quad(6.8)$$

as $k\to\infty$. As a result, we get

$$-\int_0^T\int_\Omega y_*\varphi\psi'\,dxdt+\int_0^T\int_\Omega\left((\vec{\eta},\nabla\varphi)_{\mathbb{R}^N}\;\rho+\,y_*\varphi\right)\psi\,dxdt$$
$$=\int_0^T\int_\Omega f\varphi\psi\,dxdt\quad\forall\,\varphi\in C_0^\infty(\Omega),\forall\,\psi\in C_0^\infty(0,T)\quad(6.9)$$

or $-\mathrm{div}\,(\rho_0\,\vec{\eta}\,)=f-y_*-y_*'$ in the sense of distributions.

Step 2. Here we show that $\vec{\eta}=A_0\,\nabla y_*$. To do so, we introduce the following scalar function

$$v(x)=(\vec{z},x)_{\mathbb{R}^N},\tag{6.10}$$

where $\vec{z}$ is a fixed element of $\mathbb{R}^N$. By the initial assumptions, we have

$$\int_0^T\int_\Omega\varphi\left(A_k\,(\nabla y_k-\nabla v)\,,\nabla y_k-\nabla v\right)_{\mathbb{R}^N}\rho\psi\,dxdt\geq 0,\;\forall\,\varphi\geq 0,\forall\,\psi\geq 0,$$

or, in view of (6.10), this inequality can be rewritten as

$$\int_0^T\int_\Omega\varphi\left(A_k\,(\nabla y_k-\vec{z}\,)\,,\nabla y_k-\vec{z}\,\right)_{\mathbb{R}^N}\rho\psi\,dxdt\geq 0.\tag{6.11}$$

Our next intention is to pass to the limit in (6.11) as $k\to\infty$ using Theorem 2. Having put in the statement of this lemma: $\vec{f_k}=A_k\nabla\,(y_k-v)$, and $g_k=y_k-v$ for all $k\in\mathbb{N}$, we see that the sequence $\{g_k=y_k-v\}_{k\in\mathbb{N}}$ satisfies all assumptions of Theorem 2. In view of (6.7) and (6.3), we have

$$\vec{f_k}=A_k\nabla\,(y_k-v)=A_k\,(\nabla y_k-\vec{z})\rightharpoonup\vec{\eta}-A_0\vec{z}\quad\text{in}\quad L^2(0,T;L^2(\Omega,\rho\,dx))^N.\tag{6.12}$$

It remains to show that the sequence $\left\{\vec{f_k}=A_k\nabla\,(y_k-v)\right\}_{k\in\mathbb{N}}$ is bounded in X_ρ. Indeed, from integral identity (6.8), we get

$$-\int_0^T\int_\Omega\mathrm{div}_\rho\,\left(A_k\,\nabla y_k\right)\varphi\rho\psi\,dxdt$$
$$=\int_0^T\int_\Omega\varphi\,(f-y_k)\,\psi\,dxdt+\int_0^T\int_\Omega y_k\varphi\psi'\,dxdt\quad\forall\,k\in\mathbb{N}.$$

Since $(f-y_k-y_k')\rightharpoonup(f-y_*-y_*')=\rho^{-1}\,(f-y_*-y_*')\,\rho$ in $L^2((0,T)\times\Omega)$, it follows that the sequence

$$\{\mathrm{div}_\rho\,(A_k\nabla y_k)\}_{k\in\mathbb{N}}\quad\text{is weakly compact in}\quad L^2(0,T;L^2(\Omega,\rho\,dx)),$$

and

$$\operatorname{div}_{\rho_k}\ (A_k \nabla y_k) \rightharpoonup \rho^{-1}\ \left(y_* + y_*' - f\right) \quad \text{in} \ \ L^2(0,T;L^2(\Omega,\rho\,dx)). \tag{6.13}$$

To apply Theorem 2 we have to show that the sequence $\{\operatorname{div}_\rho \left(A_k\ \vec{z}\right)\}_{k\in\mathbb{N}}$ is also weakly convergent in $L^2(0,T;L^2(\Omega,\rho\,dx))$, where the elements $\operatorname{div}_\rho \left(A_k\ \vec{z}\right)$ are defined as

$$\int_\Omega \left(A_k\ \vec{z}, \nabla\varphi\right)_{\mathbb{R}^N} \rho\,dx = -\int_\Omega \varphi \operatorname{div}_\rho \left(A_k\ \vec{z}\right)\ \rho\,dx\ \forall\,\varphi \in C_0^\infty(\Omega),\ \forall\,k\in\mathbb{N}.$$

Indeed, for every test function $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\int\limits_\Omega \left(A_k\ \vec{z}, \nabla\varphi\right)_{\mathbb{R}^N} \rho\,dx &= \int\limits_\Omega \left(\begin{bmatrix} (\vec{a}_{1\,k}(x), \vec{z})_{\mathbb{R}^N} \\ \dots \\ (\vec{a}_{n\,k}(x), \vec{z})_{\mathbb{R}^N} \end{bmatrix}, \nabla\varphi \right)_{\mathbb{R}^N} \rho\,dx \\ &= \int\limits_\Omega \sum_{i=1}^N (\vec{a}_{i\,k}(x), \vec{z})_{\mathbb{R}^N} \frac{\partial\varphi}{\partial x_i} \rho\,dx = \int\limits_\Omega \sum_{i=1}^N \sum_{j=1}^N a_{i\,j}^k(x) \frac{\partial\varphi}{\partial x_i} z_j \rho\,dx = \\ &= \sum_{j=1}^N z_j \int\limits_\Omega (\vec{a}_{j\,k}(x), \nabla\varphi)_{\mathbb{R}^N} \rho\,dx = -\sum_{j=1}^N z_j \int\limits_\Omega \varphi \operatorname{div}_\rho \vec{a}_{j\,k}\,\rho\,dx = J_k. \end{aligned} \tag{6.14}$$

Then using (6.1), we get

$$\lim_{k\to\infty} J_k = -\sum_{j=1}^N z_j \lim_{k\to\infty} \int\limits_\Omega \varphi \operatorname{div}_\rho \vec{a}_{j\,k}\,\rho\,dx = -\sum_{j=1}^N z_j \int\limits_\Omega \varphi \operatorname{div}_\rho \vec{a}_{j\,0}\,\rho\,dx. \tag{6.15}$$

Applying the converse transformations with (6.15) as we did it in (6.14), we arrive at

$$\begin{aligned}\lim_{k\to\infty} \int_\Omega \varphi \operatorname{div}_\rho \left(A_k\ \vec{z}\right)\ \rho\,dx &= -\lim_{k\to\infty} \int\limits_\Omega \left(A_k\ \vec{z}, \nabla\varphi\right)_{\mathbb{R}^N} \rho\,dx \\ &= -\int\limits_\Omega \left(A_0\ \vec{z}, \nabla\varphi\right)_{\mathbb{R}^N} \rho\,dx = \int_\Omega \varphi \operatorname{div}_\rho \left(A_0\ \vec{z}\right)\ \rho\,dx \quad \forall\,\varphi\in C_0^\infty(\Omega). \end{aligned} \tag{6.16}$$

Thus, from (6.13) and (6.16) it finally follows that

$$\begin{aligned}\operatorname{div}_\rho\ (A_k\,(\nabla y_k - \vec{z}\,)) \rightharpoonup \rho^{-1}(y_* + y_*' - f) - \operatorname{div}_\rho \left(A_0\ \vec{z}\right) \\ \text{in} \ \ L^2(0,T;L^2(\Omega,\rho\,dx)). \end{aligned} \tag{6.17}$$

As a result, combining properties (6.7), (6.17), (6.12) and the fact that

$$\nabla(y_k - v) \rightharpoonup \nabla(y_* - v) \quad \text{in} \ \ L^2(0,T;L^2(\Omega,\rho\,dx))^N,$$

we see that all suppositions of Theorem 2 are fulfilled. So, passing to the limit in inequality (6.11) as $k\to\infty$, we get

$$\int_0^T \int\limits_\Omega \varphi(x)\,(\vec{\eta} - A_0\vec{z}, \nabla y_* - \vec{z}\,)_{\mathbb{R}^N}\, \rho\psi\,dxdt \geq 0, \quad \forall \vec{z}\in\mathbb{R}^N$$

for all positive $\varphi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(0,T)$. After localization, we have

$$\rho_0 \left(\vec{\eta} - A_0 \, \vec{z}, \nabla y_* - \vec{z}\right)_{\mathbb{R}^N} \geq 0, \quad \forall \vec{z} \in \mathbb{R}^N.$$

Hence

$$\vec{\eta} = A_0 \nabla y_* \quad \rho\text{-almost everywhere in } \ (0,T) \times \Omega. \tag{6.18}$$

Step 3. Taking (6.18) into account, we can represent the integral identity (6.9) in the form

$$-\int_0^T \int_\Omega y_* \varphi \psi' \, dx dt + \int_0^T \int_\Omega \left(\left(A_0 \, \nabla y_*, \nabla \varphi\right)_{\mathbb{R}^N} \rho + \, y_* \, \varphi \right) \psi \, dx dt$$
$$= \int_0^T \int_\Omega f \varphi \psi \, dx dt \quad \forall \, \varphi \in C_0^\infty(\Omega), \forall \, \psi \in C_0^\infty(0,T), \tag{6.19}$$

or $y_*' - \operatorname{div}\, (\rho \, A_0 \nabla y_* \,) + y_* = f$ in the sense of distributions. Since $C_0^\infty(\Omega)$ dense in H_ρ, this relation remains true for all $\varphi \in H_\rho$. Hence, taking $\varphi\psi = y_*$ as a test function in (6.19), we arrive to the energy equality

$$\frac{1}{2} \int_\Omega y_*^2 \, dx \Big|_0^T + \int_0^T \int_\Omega \left(\left(A_0 \, \nabla y_*, \nabla y_*\right)_{\mathbb{R}^N} \rho + \, y_0^2 \right) \, dx dt = \int_0^T \int_\Omega f y_* \, dx dt.$$

In order to conclude the proof it remans to pass to the limit in the equality

$$\int_\Omega y_0 \varphi \, dx = \lim_{t \to +0} \int_\Omega y_k \varphi \, dx$$

which holds true for all $k \in \mathbb{N}$. As a result, using (6.4), we obtain

$$\lim_{t \to +0} \int_\Omega y_* \varphi \, dx = \int_\Omega y_0 \varphi \, dx.$$

Thus the τ-limit pair (B_0, y_*) belongs to set Ξ_H, and this concludes the proof. □

Now we are in a position to state the existence of H-optimal pairs to the problem (3.6)–(3.7),(3.10).

Theorem 4. *Let ρ be a degenerate weight in the sense of Definition 1 satisfying the conditions* (3.1). *Let also $f \in L^2((0,T) \times \Omega)$ and $y_d \in L^2(\Omega)$ be given functions. Then the optimal control problem* (3.6)–(3.7), (3.10) *admits at least one H-solution*

$$(B^{opt}, y^{opt}) \in \Xi_H \subset L^1(\Omega; \mathbb{R}^{N \times N}) \times \mathcal{W}.$$

Proof. First of all we note that for the given function $f \in L^2(\Omega)$ and every admissible control $B = A\rho \in \mathcal{B}_{ad}$, there exists an H-solution $y = y(B,f) \in H_\rho$ such that energy equality (4.3) holds true. Let $\{(B_k, y_k) = (A_k \rho, y_k) \in \Xi_H\}_{k \in \mathbb{N}}$ be an H-minimizing sequence to the problem (3.6)–(3.7),(3.10). Then as follows from the inequality

$$\inf_{(B,y) \in \Xi_H} I(B,y) = \lim_{k \to \infty} \left[\zeta \int_0^T \int_\Omega |y_k(t,x) - y_d(t,x)|^2 \, dx dt \right.$$
$$\left. + \int_0^T \int_\Omega |\nabla y_k(t,x)|^2_{\mathbb{R}^N} \rho \, dx dt + \|A_k\|_{L^\infty(\Omega, R^{N \times N})} \right] < +\infty, \tag{6.20}$$

there is a constant $C > 0$ such that

$$\sup_{k\in\mathbb{N}} \|y_k\|_{L^2(\Omega)} \leq C, \quad \sup_{k\in\mathbb{N}} \|\nabla y_k\|_{L^2(\Omega,\rho\,dx)^N} \leq C.$$

Hence, in view of the definition of the class of admissible controls $\mathcal{B}_{ad}$, we may assume that, within a subsequence, there exist a distribution $y^* \in \mathcal{W}$ and a matrix $A^* \in L^\infty(\Omega;\mathbb{R}^{N\times N})$ such that

$$\begin{gathered} A_k \stackrel{*}{\rightharpoonup} A^* \text{ in } L^\infty(\Omega, R^{N\times N}), \quad y_k \rightharpoonup y^* \text{ in } L^2(0,T;H_\rho), \\ y_k' \rightharpoonup (y^*)' \text{ in } L^2(0,T;H_\rho^{-1}). \end{gathered} \tag{6.21}$$

Using the arguments of the proof of Theorem 3, it can be shown that the matrix $B^* = A^*\rho \in L^1(\Omega, R^{N\times N})$ is admissible control to the problem (3.6)–(3.7),(3.10). As a result, the pair (B^*, y^*) is the τ-limit of the H-minimizing sequence

$$\{(B_k, y_k) \in \Xi_H\}_{k\in\mathbb{N}}.$$

Then, by Theorem 3, this pair is an H-admissible to the problem (3.6)–(3.7), (3.10). Since the cost functional I is lower τ-semicontinuous, we get

$$I(B^*, y^*) \leq \liminf_{k\to\infty} I(B_k, y_k) = \inf_{(B,y)\in\Xi_H} I(B, y).$$

Hence (B^*, y^*) is an H-optimal pair, and this concludes the proof. □

References

1. Balanenko I. G., Kogut P. I. On the existence of weak optimal BV-controls in coefficients for linear elliptic problems// Visnyk DNU, Series: Mathematical Modelling, Dnipropetrovsk : DNU, **1**:8(2009), 93–103.
2. *Boccardo L., Gallouët T., Marcellini, P.* Anisotropic equations in L^1// Diff. Integral Eq., **9**(1996), 209—212.
3. *Bouchitte G., Buttazzo G.* Characterization of optimal shapes and masses through Monge- Kantorovich equation// J. Eur. Math. Soc., **3**(2001), 139–168.
4. *Briane M., Casado-Díaz J.* Two-dimensional div-curl results. Application to the lack of nonlocal effects in homogenization// Com. Part. Diff. Equa., no.4–6, **32**(2007), 935—969.
5. *Buttazzo G., Varchon N.* On the optimal reinforcement of an elastic membrane// Riv. Mat. Univ. Parma, No. 7 , **4**(2005), 115–125.
6. *Caldiroli P., Musina R.* On a variational degenerate elliptic problem// Nonlinear Diff. Equa. Appl., **7**(2000), 187–199.
7. *Chiadó Piat V., Serra Cassano F.* Some remarks about the density of smooth functions in weighted Sobolev spaces// J. Convex Analysis, No. 2, **1**(1994), 135–142.
8. *Dautray R., Lions J.-L.* Mathematical analysis and numerical methods for science and technology. Physical origins and classical methods, Vol. 1, *Springer-Verlag*, Berlin, 1985.
9. *Drabec P., Kufner A., Nicolosi N.* Nonlinear elliptic equations, singular and degenerate cases.— University of West Bohemia, 1996.
10. *Kinderlerer D., Stampacchia G.* An Introduction to Variational Inequalities and Their Applications. *Acad. Press*, New York, London, 1980.

11. *Kovalevsky A. A., Gorban Yu. S.* Degenerate anisotropic variational inequalities with L^1-data// Preprint of IAMM of NAS, Donetsk, Ukraine, 2007.
12. *Kufner A.* Weighted Sobolev Spaces. Band 31,*Teubner–Texte Math.*, Leipzig, 1980.
13. *Lions J.-L.* Quelques méthodes de résolution des problèmes aux limites non linéaire.— Dunon: Paris, 1969. — 587p.
14. *Murat F.* Un contre-exemple pour le prolème de contrôle dans les coefficients// C.R.A.S. Paris, Sér. A **273**(1971), 708-–711.
15. *Murat F.* Compacité par compensation// *Ann. Sc. Norm. Sup. Pisa*, **5**(1978), 489–507.
16. *Murthy M. K. V., Stampacchia G.* Boundary problems for some degenerate elliptic operators// Ann. Mat. Pura e Appl., **5**:4(1968), 1–122.
17. *Pastukhova S. E.* Degenerate equations of monotone type: Lavrent'ev phenomenon and attainability problems// Sbornik: Mathematics, **198**:10(2007), 1465—1494.
18. *Zeider E.* Nonlinear Analysis and its Applications, II A and II B. — New York–Heidlberg: Springer-Verlag, 1990. — 386 p.
19. *Zhikov V. V.* On Lavrentiev phenomenon// Russian J. Math. Phys., No. 2, **3**(1994), 249–269.
20. *Zhikov V. V.* Weighted Sobolev spaces// Sbornik: Mathematics, No. 8, **189**(1998), 27–58.
21. *Zhikov V. V., Pastukhova S. E.* Homogenization of degenerate elliptic equations// Siberian Math. Journal, No. 1, **49**(2006), 80–101.

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