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ON REPRESENTATION OF THE SOLUTION TO THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION IN A CIRCLE

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It has been proven that the solution to the Dirichlet problem in a circle, where the boundary is specified as $F_2(x_1, x_2) = 0$, $F_2(x_1, x_2)$ being a polynomial of degree 2, and the boundary function is specified as $Q_m(x_1, x_2)$, $Q_m(x_1, x_2)$ being a polynomial of degree m, admits representation $u(x_1, x_2) = F_2(x_1, x_2)\, P_{m-2}(x_1, x_2) + Q_m(x_1, x_2)$, where $P_{m-2}(x_1, x_2)$ is uniquely determined polynomial of degree $m-2$.

Key words: Dirichlet problem, Poisson integral, harmonic polynomials.

1. Introduction

Consider the well known Dirichlet problem for the Laplace equation in bounded simply connected domain $\mathcal{D}$ in a plane parametrized by cartesian coordinates

$$\begin{cases} \Delta u(\boldsymbol{x}) = 0\,, & \boldsymbol{x} = (x_1, x_2) \in \mathcal{D} \subset \mathbb{R}^2\,, \\ u(\boldsymbol{x}) = g(\boldsymbol{x})\,, & \boldsymbol{x} = (x_1, x_2) \in \mathcal{S} := \partial\mathcal{D}\,. \end{cases} \tag{1.1}$$

If boundary $\mathcal{S}$ is piecewise smooth, boundary function $g(\boldsymbol{x}) \in \mathscr{C}(\mathcal{S})$, then the solution to the problem (1.1) exists, is unique, and $u(\boldsymbol{x}) \in \mathscr{C}^2(\mathcal{D}) \bigcap \mathscr{C}(\overline{\mathcal{D}})$ (moreover, $u(\boldsymbol{x}) \in \mathscr{C}^\infty(\mathcal{D})$, and even $u(\boldsymbol{x}) \in \mathscr{C}^a(\mathcal{D})$) [3,4].

We apply sequentially two restrictions of the formulation of the Dirichlet problem (1.1). The first restriction is in choosing circle $\mathcal{B}_a(\boldsymbol{x}_0)$ of radius a centered at point $\boldsymbol{x}_0$ as domain $\mathcal{D}$. The problem

$$\begin{cases} \Delta u(\boldsymbol{x}) = 0\,, & \boldsymbol{x} \in \mathcal{B}_a(\boldsymbol{x}_0) := \left\{\boldsymbol{x} : |\boldsymbol{x} - \boldsymbol{x}_0|^2 < a^2\right\}, \\ u(\boldsymbol{x}) = g(\boldsymbol{x})\,, & \boldsymbol{x} \in \mathcal{S}_a(\boldsymbol{x}_0) := \left\{\boldsymbol{x} : |\boldsymbol{x} - \boldsymbol{x}_0|^2 = a^2\right\}, \end{cases} \tag{1.2}$$

is known to be an amazing example of the dependence existing between trigonometric series, harmonic functions $u(x_1, x_2)$ of real variables (x_1, x_2) and analytic functions $f(z) = u(z) + iv(z)$ of complex variable $z = x_1 + ix_2$ [2,6].

The solution to the problem (1.2) can be obtained using quite different approaches, for example, separation of polar coordinates (r, φ)

$$\begin{cases} x_1 - x_{1,0} = r\cos\varphi\,, \\ x_2 - x_{2,0} = r\sin\varphi\ , \end{cases} \qquad (r,\varphi) \in \overline{\mathcal{B}_a(\boldsymbol{x}_0)}\,, \tag{1.3}$$

applied to the Laplace equation $\Delta u = 0$ and searching for the solution u as trigonometric series [5] (the circle over the function name indicates changing polar coordinates as independent variables to cartesian ones)

$$\mathring{u}(r,\varphi) = \frac{a_0}{2} + \sum_{\mu=0}^{\infty} \left(\frac{r}{a}\right)^{\mu} \left(a_\mu \cos(\mu\varphi) + b_\mu \cos(\mu\varphi)\right), \tag{1.4}$$

where a_0, a_μ, b_μ are the Fourier coefficients ($\mu \in \mathbb{N}$) [6]

$$\begin{cases} a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \widetilde{g}(\psi)\,\mathrm{d}\psi\,, \quad a_\mu = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \widetilde{g}(\psi)\cos(\mu\psi)\,\mathrm{d}\psi\,, \\ \qquad\qquad\qquad\qquad\quad b_\mu = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \widetilde{g}(\psi)\sin(\mu\psi)\,\mathrm{d}\psi\,, \end{cases} \tag{1.5}$$

for the boundary function $\widetilde{g}(\varphi) := g(\boldsymbol{x})|_{\boldsymbol{x}\in\mathcal{S}_a(\boldsymbol{x}_0)} = g(x_{1,0}+a\cos\varphi,\ x_{2,0}+a\sin\varphi)$.

Summing the series (1.4) over μ gets the Poisson integral (formula) [3]

$$\mathring{u}(r,\varphi) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{\widetilde{g}(\psi)\,\mathrm{d}\psi}{a^2 - 2ar\cos(\psi-\varphi) + r^2}\,, \tag{1.6}$$

that can be treated in terms of convolution of the boundary function $\widetilde{g}(\varphi)$ and the Poisson kernel. The Poisson integral (1.6) can be obtained as well for the real and imaginary parts of function $f(z) = u+iv$ analytic in $\mathcal{B}_a(\boldsymbol{x}_0)$ using the Cauchy integral (formula) [2]

$$f(z) = \frac{1}{2\pi i} \oint\limits_{|\zeta - z_0| = 0} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}\,, \tag{1.7}$$

where $z_0 = x_{1,0} + ix_{2,0}$, and equation $|\zeta - z_0| = 0$ specifies $\mathcal{S}_a(\boldsymbol{x}_0)$.

The second restriction of the formulation of the Dirichlet problem is in choosing a polynomial of degree m as the boundary function

$$Q_m(\boldsymbol{x}) = \sum_{|\alpha|=0}^{m} a_\alpha\,\boldsymbol{x}^\alpha = \sum_{p+q=0}^{m} a_{p,q}\,x_1^p\,x_2^q\,, \tag{1.8}$$

where $\alpha = (p,q)$ is multi-index, $p,q \geqslant 0$, $p,q \in \mathbb{Z}$; $|\alpha| = p+q$; $a_\alpha = a_{p,q} \in \mathbb{R}$.

For the boundary function $g(\boldsymbol{x}) = Q_m(\boldsymbol{x})$ the Fourier coefficients (1.5) with numbers $\mu > m$ equal zero. This means that the solution to the Dirichlet problem

$$\begin{cases} \Delta u(\boldsymbol{x}) = 0\,, & \boldsymbol{x} \in \mathcal{B}_a(\boldsymbol{x}_0)\,, \\ u(\boldsymbol{x}) = Q_m(\boldsymbol{x})\,, & \boldsymbol{x} \in \mathcal{S}_a(\boldsymbol{x}_0)\,, \end{cases} \tag{1.9}$$

written in polar coordinates (r, φ) is a finite series (1.4), whereas written in cartesian coordinates (x_1, x_2) it is a polynomial of degree m.

Proposition 1.1. Solution to the Dirichlet problem (1.9), where the boundary function is a polynomial $Q_m(\boldsymbol{x})$ (1.8), admits the following representation

$$u(\boldsymbol{x}) = F_2(\boldsymbol{x})\, P_{m-2}(\boldsymbol{x}) + Q_m(\boldsymbol{x})\,, \tag{1.10}$$

where the polynomial of second degree $F_2(\boldsymbol{x})$ specifies the boundary $\mathcal{S}_a(\boldsymbol{x}_0)$

$$F_2(\boldsymbol{x}) = |\boldsymbol{x} - \boldsymbol{x}_0|^2 - a^2 = 0\,,$$

and $P_{m-2}(\boldsymbol{x})$ is uniquely determined polynomial of degree $m - 2$.

The article is arranged as follows. Proposition 1.1 is proved in Section 2; some examples of representation (1.10) for the solution to the Dirichlet problem (1.9), including nontrivial ones, are present in Section 3. No application or extension of representation (1.10), for example, in the case of the Dirichlet problem in a ball $\subset \mathbb{R}^3$, is discussed.

2. Proving the representation

In this Section we change the independent variables $\boldsymbol{x} \to \boldsymbol{y}$, setting $\boldsymbol{x} = \boldsymbol{y}+\boldsymbol{x}_0$, and replace the Dirichlet problem (1.9) with the following one

$$\begin{cases} \Delta w(\boldsymbol{y}) = 0\,, & \boldsymbol{y} \in \mathcal{B}_a(\boldsymbol{0})\,, \\ w(\boldsymbol{y}) = R_m(\boldsymbol{y})\,, & \boldsymbol{y} \in \mathcal{S}_a(\boldsymbol{0})\,, \end{cases} \tag{2.1}$$

where $w(\boldsymbol{y}) := u(\boldsymbol{y} + \boldsymbol{x}_0)$,

$$R_m(\boldsymbol{y}) := Q_m(\boldsymbol{y}+\boldsymbol{x}_0) = \sum_{|\alpha|=0}^{m} a_\alpha\, (\boldsymbol{y}+\boldsymbol{x}_0)^\alpha = \sum_{|\alpha|=0}^{m} b_\alpha\, \boldsymbol{y}^\alpha = \sum_{p+q=0}^{m} b_{p,q}\, y_1^p\, y_2^q\,, \tag{2.2}$$

and as a consequence representation (1.10) is replaced with the following one

$$w(\boldsymbol{y}) = G_2(\boldsymbol{y})\, S_{m-2}(\boldsymbol{y}) + R_m(\boldsymbol{y})\,, \tag{2.3}$$

where $G_2(\boldsymbol{y}) := F_2(\boldsymbol{y}+\boldsymbol{x}_0) = |\boldsymbol{y}|^2 - a^2$, $S_{m-2}(\boldsymbol{y}) := P_{m-2}(\boldsymbol{y}+\boldsymbol{x}_0)$.

Then we formulate and prove some auxiliary propositions concerning representation (2.3) for the Dirichlet problem (2.1), and finally prove main Proposition 1.1. To prove the auxiliary propositions we use well known formulas [1] for powers of trigonometric functions $\cos\varphi$ and $\sin\varphi$ in terms of these functions of multiples of the argument

$$\begin{cases} \cos^m\varphi = \dfrac{1}{2^{m-1}} \displaystyle\sum_{\mu=0}^{\frac{m-1}{2}} C_m^\mu \cos\left[(m-2\mu)\varphi\right], \\ \sin^m\varphi = \dfrac{1}{2^{m-1}} \displaystyle\sum_{\mu=0}^{\frac{m-1}{2}} C_m^\mu \sin\left[(m-2\mu)\varphi\right](-1)^{\frac{m-1}{2}-\mu}, \end{cases} \tag{2.4}$$

when m is an odd integer, and

$$\begin{cases} \cos^m\varphi = \dfrac{1}{2^m} C_m^{\frac{m}{2}} + \dfrac{1}{2^{m-1}} \displaystyle\sum_{\mu=0}^{\frac{m}{2}-1} C_m^\mu \cos\left[(m-2\mu)\varphi\right], \\ \sin^m\varphi = \dfrac{1}{2^m} C_m^{\frac{m}{2}} + \dfrac{1}{2^{m-1}} \displaystyle\sum_{\mu=0}^{\frac{m}{2}-1} C_m^\mu \cos\left[(m-2\mu)\varphi\right](-1)^{\frac{m}{2}-\mu}, \end{cases} \tag{2.5}$$

when m is an even integer, and formulas [1] for trigonometric functions $\cos\varphi$ and $\sin\varphi$ of multiples of the argument in terms of powers of these functions

$$\begin{cases} \cos m\varphi = C_m^0 \cos^m\varphi - C_m^2 \cos^{m-2}\varphi \sin^2\varphi + \ldots, \\ \sin m\varphi = C_m^1 \cos^{m-1}\varphi \sin\varphi - C_m^3 \cos^{m-3}\varphi \sin^3\varphi + \ldots \end{cases} \tag{2.6}$$

Expanding powers of trigonometric functions $\cos\varphi$ and $\sin\varphi$ into Fourier series after formulas (1.5) leads to formulas (2.4) and (2.5), therefore we use the both latter as the corresponding Fourier series.

Note also that when replacing trigonometric functions $\cos\varphi$ and $\sin\varphi$ on the right hand sides of formulas (2.6) with x_1 and x_2 respectively, one gets harmonic polynomials $u(x_1,x_2) = \mathtt{Re}\, z^m$ and $v(x_1,x_2) = \mathtt{Im}\, z^m$.

Proposition 2.1. The solution to the Dirichlet problem (2.1), where the boundary function is $R_m(\boldsymbol{y}) = y_1^m$, $m \geqslant 2$, $m \in \mathbb{N}$, admits representation (2.3).

Proof. We consider two possible cases of power m being odd or even integers separately.

a) Let m be an odd integer. Represent the boundary function on $\mathcal{S}_a(\mathbf{0})$ and in the closure of $\mathcal{B}_a(\mathbf{0})$ using formulas (2.4):

$$R_m(\boldsymbol{y})\big|_{\boldsymbol{y}\in\mathcal{S}_a(\mathbf{0})} = a^m \cos^m\varphi =$$
$$= \frac{a^m}{2^{m-1}} \sum_{\mu=0}^{\frac{m-1}{2}} C_m^\mu \cos\left[(m-2\mu)\varphi\right] =: \widetilde{R}_m(\varphi)\,, \tag{2.7}$$

$$R_m(\boldsymbol{y})\big|_{\boldsymbol{y}\in\overline{\mathcal{B}_a(\mathbf{0})}} = r^m \cos^m\varphi =$$
$$= \frac{r^m}{2^{m-1}} \sum_{\mu=0}^{\frac{m-1}{2}} C_m^\mu \cos\left[(m-2\mu)\varphi\right] =: \mathring{R}_m(r,\varphi)\,; \tag{2.8}$$

here and below the tilde and the circle over function name indicate that the polar angle φ and the polar radius r and angle φ are used as independent variables on the boundary and in the domain respectively (as in Section 1).

We use expansion of the boundary function into Fourier series (2.7) to write the solution to the Dirichlet problem (2.1) in polar coordinates ($(r,\varphi)\in\overline{\mathcal{B}_a(\mathbf{0})}$) as follows

$$\mathring{w}(r,\varphi) = \frac{a^m}{2^{m-1}} \sum_{\mu=0}^{\frac{m-1}{2}} C_m^\mu \left(\frac{r}{a}\right)^{m-2\mu} \cos\left[(m-2\mu)\varphi\right]. \tag{2.9}$$

Then we transform expression (2.9) identically, explicitly separating the boundary function $\mathring{R}_m(r,\varphi)$ (2.8) to obtain

$$\mathring{w}(r,\varphi) = \mathring{w}(r,\varphi) \mp \mathring{R}_m(r,\varphi) =$$
$$= \frac{1}{2^{m-1}} \sum_{\mu=1}^{\frac{m-1}{2}} C_m^\mu\, r^{m-2\mu} \left(a^{2\mu}-r^{2\mu}\right) \cos\left[(m-2\mu)\varphi\right] + \mathring{R}_m(r,\varphi)\,. \tag{2.10}$$

Represent the binomials in the parentheses of expression (2.10) as follows: $a^{2\mu}-r^{2\mu} = \left(a^2-r^2\right) A_\mu(r)$, where $A_\mu(r)$ are polynomials of degree $2\mu-2$ in r

$$A_{2\mu-2}(r) = \begin{cases} 1\,, & \mu = 1\,, \\ r^{2\mu-2} + a^2 r^{2\mu-4} + \ldots + a^{2\mu-4} r^2 + a^{2\mu-2}\,, & \mu > 1\,, \end{cases} \tag{2.11}$$

then expression (2.10) reads

$$\mathring{w}(r,\varphi) = \frac{a^2-r^2}{2^{m-1}} \sum_{\mu=1}^{\frac{m-1}{2}} C_m^\mu\, A_{2\mu-2}(r)\, r^{m-2\mu} \cos\left[(m-2\mu)\varphi\right] + \mathring{R}_m(r,\varphi)\,. \tag{2.12}$$

Now we apply formula (2.6) for trigonometric function $\cos\varphi$ of multiples of the argument in expression (2.12)

$$\cos\left[(m-2\mu)\varphi\right] = \cos^{m-2\mu}\varphi - C^2_{m-2\mu}\cos^{m-2\mu-2}\varphi\,\sin^2\varphi\, + \ldots$$

Homogenious trigonometric monomials $\cos^p\varphi\,\sin^q\varphi$ of degree $p+q=m-2\mu$, $p,q \geqslant 0$, $p,q\in\mathbb{Z}$, being present on the right hand side of the expression above after multiplication by $r^{m-2\mu}$ produce homogenius monomials $y_1^p\,y_2^q$.

The least degree of monomials $y_1^p\,y_2^q$ equals $(m-2\mu)|_{2\mu=m-1}=1$, when $A_\mu(r)$ defined by expression (2.11) is a polynomial of zero ($m=3$) or even ($m>3$) degree $(2\mu-2)|_{2\mu=m-1}=m-3$ in r. Therefore, in this case the corresponding term in expression (2.12) is a polynomial of degree $m-2$ in variables y_1, y_2.

The highest degree of monomials $y_1^p\,y_2^q$ equals $(m-2\mu)|_{\mu=1}=m-2$, when $A_\mu(r)$ defined by expression (2.11) is a polynomial of degree $(2\mu-2)|_{\mu=1}=0$ in r. Therefore, in this case the corresponding term in expression (2.12) is a polynomial of degree $m-2$ in variables y_1, y_2 as well.

This means, that all the terms in the sum of expression (2.12) are polynomials of degree $m-2$ in variables y_1, y_2, therefore, the sum itself is a polynomial of the same degree.

To complete transformation of expression (2.12) to representation (2.3) there remain to replace r^2 preceding the sum with $\boldsymbol{y}^2=y_1^2+y_2^2$ and change the independent variables in boundary function $\mathring{R}_m(r,\varphi)$ back according to (2.8).

b) Let m be an even integer. Represent the boundary function on $\mathcal{S}_a(\mathbf{0})$ and in the closure of $\mathcal{B}_a(\mathbf{0})$ using formulas (2.5):

$$\widetilde{R}_m(\varphi) := \frac{a^m}{2^m}\,C_m^{\frac{m}{2}} + \frac{a^m}{2^{m-1}}\sum_{\mu=0}^{\frac{m}{2}-1} C_m^\mu \cos\left[(m-2\mu)\varphi\right], \tag{2.13}$$

$$\mathring{R}_m(r,\varphi) := \frac{r^m}{2^m}\,C_m^{\frac{m}{2}} + \frac{r^m}{2^{m-1}}\sum_{\mu=0}^{\frac{m}{2}-1} C_m^\mu \cos\left[(m-2\mu)\varphi\right]. \tag{2.14}$$

We use expansion of the boundary function into Fourier series (2.13) to write the solution to the Dirichlet problem (2.1) in polar coordinates ($(r,\varphi)\in\overline{\mathcal{B}_a(\mathbf{0})}$) as follows

$$\mathring{w}(r,\varphi) = \frac{a^m}{2^m}\,C_m^{\frac{m}{2}} + \frac{a^m}{2^{m-1}}\sum_{\mu=0}^{\frac{m}{2}-1} C_m^\mu \left(\frac{r}{a}\right)^{m-2\mu} \cos\left[(m-2\mu)\varphi\right]. \tag{2.15}$$

Then we transform expression (2.15) identically

$$\mathring{w}(r,\varphi) = \frac{1}{2^m} C_m^{\frac{m}{2}} \left(a^m - r^m\right) + $$
$$+ \frac{1}{2^{m-1}} \sum_{\mu=1}^{\frac{m}{2}-1} C_m^{\mu} r^{m-2\mu} \left(a^{2\mu} - r^{2\mu}\right) \cos\left[(m-2\mu)\varphi\right] + \mathring{R}_m(r,\varphi)$$

and rearrange the terms to obtain

$$\mathring{w}(r,\varphi) = \frac{a^2 - r^2}{2^m} C_m^{\frac{m}{2}} B_{m-2}(r) + $$
$$+ \frac{a^2 - r^2}{2^{m-1}} \sum_{\mu=1}^{\frac{m}{2}-1} C_m^{\mu} A_{2\mu-2}(r)\, r^{m-2\mu} \cos\left[(m-2\mu)\varphi\right] + \mathring{R}_m(r,\varphi)\,,$$

where $A_{2\mu-2}(r)$ are the polynomials defined above by expression (2.11); $B_{m-2}(r)$ is the polynomial of degree $m-2$ in r defined as

$$B_{m-2}(r) = \begin{cases} 1\,, & m = 2\,, \\ r^{m-2} + a^2 r^{m-4} + \ldots + a^{m-4} r^2 + a^{m-2}\,, & m > 2\,. \end{cases}$$

The remaining part of the proof is similar to that in the case of m being an odd integer. □

Proposition 2.2. The solution to the Dirichlet problem (2.1), where the boundary function is $R_m(\boldsymbol{y}) = y_2^m$, $m \geqslant 2$, $m \in \mathbb{N}$, admits representation (2.3).

Proof. It is enough to repeat the proof of Proposition 2.1 if one replaces formulas (2.4), (2.5), and (2.6) for $\cos\varphi$ with the same formulas for $\sin\varphi$. □

Proposition 2.3. The solution to the Dirichlet problem (2.1), where the boundary function is $R_m(\boldsymbol{y}) = y_1^p\, y_2^q$, $p + q = m \geqslant 2$, $p, q \in \mathbb{Z}_+$, admits representation (2.3).

Proof. We consider two possible cases of power m being odd or even integers separately.

a) Let m be an odd integer; it can be represented as the sum of an odd and an even integers. Therefore, this case admits two subcases: 1) p is an odd integer, q is an even integer; 2) p is an even integer, q is an odd integer.

b) Let m be an even integer; it can be represented as the sum of both odd integers or both even integers. Therefore, this case admits two subcases: 3) p is an odd integer, q is an odd integer; 4) p is an even integer, q is an even integer.

Consider subcase 1) (keeping in mind that $p \geqslant 1$, $q \geqslant 2$, $m = p + q \geqslant 3$) and represent the boundary function on $\mathcal{S}_a(\mathbf{0})$ using formulas (2.4), (2.5)

$$
\begin{aligned}
R_m(\boldsymbol{y})\big|_{\boldsymbol{y}\in\mathcal{S}_a(\mathbf{0})} &= a^p \cos^p\varphi \; a^q \sin^q\varphi = \\
&= a^m \left(\frac{1}{2^{p-1}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \cos\left[(p-2\mu)\varphi\right] \right) \times \\
&\quad \times \left(\frac{1}{2^q} C_q^{\frac{q}{2}} + \frac{1}{2^{q-1}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma} C_q^\gamma \cos\left[(q-2\gamma)\varphi\right] \right) = \\
&= \frac{a^m}{2^{m-1}} C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \cos\left[(p-2\mu)\varphi\right] + \\
&+ \frac{a^m}{2^{m-2}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma} C_p^\mu C_q^\gamma \cos\left[(p-2\mu)\varphi\right] \cos\left[(q-2\gamma)\varphi\right].
\end{aligned}
$$

We apply trigonometric formula

$$
2\cos\varphi_1 \cos\varphi_2 = \cos(\varphi_1 - \varphi_2) + \cos(\varphi_1 + \varphi_2)
$$

to the second term of the latter, therefore representation of the boundary function on $\mathcal{S}_a(\mathbf{0})$ above reads

$$
\left\{
\begin{aligned}
\widetilde{R}_m(\varphi) &= \frac{a^m}{2^{m-1}} C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu \cos\left[(p-2\mu)\varphi\right] + \\
&+ \frac{a^m}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma} C_p^\mu C_q^\gamma \cos\left[(p+q-2\mu-2\gamma)\varphi\right] + \\
&+ \frac{a^m}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma} C_p^\mu C_q^\gamma \cos\left[(p-q-2\mu+2\gamma)\varphi\right].
\end{aligned}
\right. \tag{2.16}
$$

Then we use representation (2.16) of the boundary function on $\mathcal{S}_a(\mathbf{0})$ to represent the boundary function in the closure of $\mathcal{B}_a(\mathbf{0})$

$$\left\{\begin{aligned} \mathring{R}_m(r,\varphi) \;=\; & \frac{r^m}{2^{m-1}}\, C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu} \cos\left[(p-2\mu)\varphi\right] + \\ + \; & \frac{r^m}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma}\, C_p^{\mu}\, C_q^{\gamma} \cos\left[(p+q-2\mu-2\gamma)\varphi\right] + \\ + \; & \frac{r^m}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma}\, C_p^{\mu}\, C_q^{\gamma} \cos\left[(p-q-2\mu+2\gamma)\varphi\right], \end{aligned}\right. \tag{2.17}$$

to write down the solution to the Dirichlet problem (2.1) in polar coordinates ($(r,\varphi) \in \overline{\mathcal{B}_a(\mathbf{0})}$) as follows

$$\begin{aligned} \mathring{w}(r,\varphi) \;=\; & \frac{a^m}{2^{m-1}}\, C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu} \left(\frac{r}{a}\right)^{p-2\mu} \cos\left[(p-2\mu)\varphi\right] + \\ + \; & \frac{a^m}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma}\, C_p^{\mu}\, C_q^{\gamma} \left(\frac{r}{a}\right)^{p+q-2\mu-2\gamma} \cos\left[(p+q-2\mu-2\gamma)\varphi\right] + \\ + \; & \frac{a^m}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma}\, C_p^{\mu}\, C_q^{\gamma} \left(\frac{r}{a}\right)^{p-q-2\mu+2\gamma} \cos\left[(p-q-2\mu+2\gamma)\varphi\right], \end{aligned}$$

and to transform the latter identically

$$\left\{\begin{aligned} \mathring{w}(r,\varphi) \;\overset{(2.17)}{=}\; & \mathring{w}(r,\varphi) \mp \mathring{R}_m(r,\varphi) = \\ = \; & \frac{1}{2^{m-1}}\, C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^{\mu}\, W_{\mu}(r) \cos\left[(p-2\mu)\varphi\right] + \\ + \; & \frac{1}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, V_{\mu,\gamma}(r) \cos\left[(p+q-2\mu-2\gamma)\varphi\right] + \\ + \; & \frac{1}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma}\, C_p^{\mu}\, C_q^{\gamma}\, U_{\mu,\gamma}(r) \cos\left[(p-q-2\mu+2\gamma)\varphi\right] + \\ + \; & \mathring{R}_m(r,\varphi), \end{aligned}\right. \tag{2.18}$$

where for the sake of shortness there introduced the following functions

$$\begin{cases} W_\mu(r) &= r^{p-2\mu}\, a^{m-p+2\mu} - r^m\,, \\ V_{\mu,\gamma}(r) &= r^{p+q-2\mu-2\gamma}\, a^{m-p-q+2\mu+2\gamma} - r^m\,, \\ U_{\mu,\gamma}(r) &= r^{p-q-2\mu+2\gamma}\, a^{m-p+q+2\mu-2\gamma} - r^m\,. \end{cases} \tag{2.19}$$

Represent functions $W_\mu(r)$ (2.19) of the first term of solution (2.18) as follows

$$W_\mu(r) = r^{p-2\mu}\left(a^{m-p+2\mu} - r^{m-p+2\mu}\right) = r^{p-2\mu}\left(a^2 - r^2\right) E_{m-(p-2\mu)-2}(r)\,, \tag{2.20}$$

where $E_{m-(p-2\mu)-2}(r)$ are polynomials of degree $k := m-(p-2\mu)-2$ (that equals zero or an even integer) in r

$$E_{m-(p-2\mu)-2}(r) = \begin{cases} 1\,, & k=0\,, \\ r^k + a^2 r^{k-2} + \ldots + a^{k-2} r^2 + a^k\,, & k>2\,. \end{cases} \tag{2.21}$$

The least degree of the polynomials equals $m-p-2$ at $\mu = 0$ (the lower limit of summation), whereas the highest degree equals $m-3$ at $2\mu = p-1$ (the upper limit of summation).

Therefore, the first term in solution (2.18) now reads as follows

$$\frac{a^2 - r^2}{2^{m-1}}\, C_q^{\frac{q}{2}} \sum_{\mu=0}^{\frac{p-1}{2}} C_p^\mu\, E_{m-(p-2\mu)-2}(r)\, r^{p-2\mu} \cos\left[(p-2\mu)\varphi\right]. \tag{2.22}$$

Trigonometric functions $\cos\left[(p-2\mu)\varphi\right]$ in the sum of expression (2.22), according to formula (2.6), are homogenious trigonometric polynomials of degree $p-2\mu$, and after multiplication by $r^{p-2\mu}$ they produce homogenious polynomials of degree $p-2\mu$ in variables y_1, y_2. Multiplicating the latter polynomials by polynomials $E_{m-(p-2\mu)-2}(r)$ (2.21) of degree $m-(p-2\mu)-2$ produces polynomials of degree $m-2$ in variables y_1, y_2.

Represent functions $V_{\mu,\gamma}(r)$ (2.19) of the second term of solution (2.18) as follows

$$\begin{aligned} V_{\mu,\gamma}(r) &= r^{m-2(\mu+\gamma)}\, a^{2(\mu+\gamma)} - r^m = \\ &= r^{m-2(\mu+\gamma)}\left(a^{2(\mu+\gamma)} - r^{2(\mu+\gamma)}\right) = \\ &= r^{m-2(\mu+\gamma)}\left(a^2 - r^2\right) F_{2(\mu+\gamma)-2}(r)\,, \end{aligned} \tag{2.23}$$

where $F_{2(\mu+\gamma)-2}(r)$ are polynomials of degree $k := 2(\mu+\gamma)-2$ (that equals zero or an even integer) in r

$$F_{2(\mu+\gamma)-2}(r) = \begin{cases} 1\,, & \mu+\gamma = 1\,, \\ r^k + a^2 r^{k-2} + \ldots + a^{k-2} r^2 + a^k\,, & \mu+\gamma > 1\,. \end{cases} \tag{2.24}$$

Representation (2.23), (2.24) for functions $V_{\mu,\gamma}(r)$ (2.19) is not applicable at two lower limits of summation ($\mu = \gamma = 0$), for in this case the corresponding term in double sum disappears ($V_{\mu,\gamma}(r) \equiv 0$).

Therefore, the second term in solution (2.18) reads as follows

$$\frac{a^2-r^2}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma}\, C_p^\mu\, C_q^\gamma\, F_{2(\mu+\gamma)-2}(r)\, r^{m-2(\mu+2\gamma)-2} \cos\left[(m-2(\mu+\gamma))\varphi\right]. \tag{2.25}$$

Trigonometric functions $\cos\left[(m-2(\mu+\gamma))\varphi\right]$ in the sum of expression (2.25), according to formula (2.6), are homogenious trigonometric polynomials of degree $m-2(\mu+\gamma)$, and after multiplication by $r^{m-2(\mu+\gamma)}$ they produce homogenious polynomials of degree $m-2(\mu+\gamma)$ in variables y_1, y_2. Multiplicating the latter polynomials by polynomials $F_{2(\mu+\gamma)-2}(r)$ (2.24) of degree $2(\mu+\gamma)-2$ produces polynomials of degree $m-2$ in variables y_1, y_2.

And finally consider functions $U_{\mu,\gamma}(r)$ (2.19) of the third term of solution (2.18) and write down powers of r and a to obtain

$$\begin{aligned} m-p+q+2\mu-2\gamma &= (p+q)-p+q+2\mu-2\gamma = 2(q+\mu-\gamma)\,, \\ p-q-2\mu+2\gamma &= (m-q)-q-2\mu+2\gamma = m-2(q+\mu-\gamma)\,. \end{aligned}$$

Now represent functions $U_{\mu,\gamma}(r)$ (2.19) as follows

$$\begin{aligned} U_{\mu,\gamma}(r) &= r^{m-2(q+\mu-\gamma)}\, a^{2(q+\mu-\gamma)} - r^m = \\ &= r^{m-2(q+\mu-\gamma)} \left(a^{2(q+\mu-\gamma)} - r^{2(q+\mu-\gamma)}\right) = \\ &= r^{m-2(q+\mu-\gamma)} \left(a^2 - r^2\right) H_{2(q+\mu-\gamma)-2}(r)\,, \end{aligned} \tag{2.26}$$

where $H_{2(q+\mu-\gamma)-2}(r)$ are polynomials of degree $k := 2(q+\mu-\gamma)-2$ (that equals zero or an even integer) in r

$$H_{2(q+\mu-\gamma)-2}(r) = \begin{cases} 1\,, & q+\mu-\gamma = 1\,, \\ r^k + a^2 r^{k-2} + \ldots + a^{k-2} r^2 + a^k\,, & q+\mu-\gamma > 1\,. \end{cases} \qquad (2.27)$$

Representation (2.26), (2.27) for functions $U_{\mu,\gamma}(r)$ (2.19) is not applicable when $q + \mu - \gamma = 0$, for in this case the corresponding term in double sum disappears $(U_{\mu,\gamma}(r) \equiv 0)$.

Therefore, the third term in solution (2.18) now reads as follows

$$\frac{a^2 - r^2}{2^{m-1}} \sum_{\mu=0}^{\frac{p-1}{2}} \sum_{\gamma=0}^{\frac{q}{2}-1} (-1)^{\frac{q}{2}-\gamma}\, C_p^\mu\, C_q^\gamma\, H_{2(q+\mu-\gamma)-2}(r) \times \\ \times\, r^{m-2(q+\mu-\gamma)} \cos\left[(m - 2(q+\mu-\gamma))\varphi\right]. \qquad (2.28)$$

Trigonometric functions $\cos\left[(m - 2(q+\mu-\gamma))\varphi\right]$ in the sum of expression (2.28), according to formula (2.6), are homogenious trigonometric polynomials of degree $m-2(q+\mu-\gamma)$, and after multiplication by $r^{m-2(q+\mu-\gamma)}$ they produce homogenious polynomials of degree $m-2(q+\mu-\gamma)$ in variables y_1, y_2. Multiplicating the latter polynomials by polynomials $H_{2(q+\mu-\gamma)-2}(r)$ (2.27) of degree $2(q+\mu-\gamma)-2$ produces polynomials of degree $m-2$ in variables y_1, y_2.

Therefore, the first three terms (2.22), (2.25), (2.28) of solution (2.18) to the Dirichlet problem (2.1), after replacing r^2 preceding the sums with $\boldsymbol{y}^2 = y_1^2 + y_2^2$, are identical to the first term of representation (2.3). There remains to change the independents variables in boundary function $\mathring{R}_m(r,\varphi)$ back according to (2.8). This completes the proof in the subcase 1).

The subcases 2), 3), and 4) are considered similarly to subcase 1) and are missed here. □

Proposition 2.4. The solution to the Dirichlet problem (2.1), where the boundary function is $R_m(\boldsymbol{y})$ (2.2), admits representation (2.3).

Proof. Proposition 2.4 holds due to: 1) linearity of polynomial $R_m(\boldsymbol{y})$ (2.2) in monomials $y_1^p\, y_2^q$, $p + q = m$, $p, q \geqslant 0$, $p, q \in \mathbb{Z}$; 2) linearity of the Dirichlet problem (2.1); 3) Propositions 2.1, 2.2, 2.3. □

Auxiliary Proposition 2.4 let us prove main Proposition 1.1. Indeed, change the independent variables back $\boldsymbol{y} \to \boldsymbol{x}$, setting $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_0$ in the Dirichlet problem (2.1) and representation (2.3). Then the Dirichlet problem (2.1) and representation (2.3) are replaced with the Dirichlet problem (1.9) and representation (1.10).

3. Examples of the representation

In this Section we consider two examples illustrating the representation.

First we consider Example 3.1 of representation (2.3) for the solution to the Dirichlet problem (2.1) in a circle $\mathcal{B}_a(\mathbf{0})$, where the boundary function is given as $Q_m(\boldsymbol{x}) = x_1^m$, $m = 2, 3, 4$ (for the sake of convenience we write $\boldsymbol{x}$, u, Q_m, as in problem (1.9), rather than $\boldsymbol{y}$, w, R_m, as in problem (2.1)). Representation (2.3) is obtained either as evident, or following the proof of Proposition 2.1 (indeed, it is the proof of representation 2.1 that follows Example 3.1 in case of $Q_4(\boldsymbol{x}) = x_1^4$). Examples similar to Example 3.1 happened to be inductive for representation (2.3) yet before Propositions 2.1, 2.2, 2.3, 2.4 were proved.

Then we consider Example 3.2 of representation (1.10) for the solution to the Dirichlet problem (1.9) in a circle $\mathcal{B}_a(\boldsymbol{x}_0)$, where the boundary function is given as $Q_m(\boldsymbol{x}) = x_1^p x_2^q$, $p, q > 0$, $m = p + q = 2, 3, 4, 5$. Representation (1.10) is obtained using division of polynomials in variables (x_1, x_2) (detailed discussion is presented only in case of $Q_5(\boldsymbol{x}) = x_1^3 x_2^2$). Division of 'large' polynomials was performed using the `MAPLE` environment. Examples similar to Example 3.2 were considered as tests for representation (1.10) before and after proving Proposition 1.1.

Example 3.1. Consider the Dirichlet problem (2.1), where the boundary of the circle is specified by equation $F_2(\boldsymbol{x}) = |\boldsymbol{x}|^2 - a^2 = x_1^2 + x_2^2 - a^2 = 0$.

a) Let the boundary function be given as $Q_2(\boldsymbol{x}) = x_1^2$. On $\mathcal{S}_a(0,0)$ the boundary function reads

$$\widetilde{Q}_2(\varphi) = a^2 \cos^2\varphi \overset{(2.5)}{=} \frac{a^2}{2} + \frac{a^2}{2} \cos 2\varphi\,,$$

and from this it immediately follows the solution to the Dirichlet problem (2.1) in polar coordinates

$$\mathring{u}(r,\varphi) = \frac{a^2}{2} + \frac{a^2}{2}\left(\frac{r}{a}\right)^2 \cos 2\varphi \overset{(2.6)}{=} \frac{a^2}{2} + \frac{r^2}{2}\left(\cos^2\varphi - \sin^2\varphi\right),$$

and then in cartesian ones

$$u(\boldsymbol{x}) = \frac{a^2}{2} + \frac{1}{2}\left(x_1^2 - x_2^2\right) = -\frac{1}{2}\left(x_1^2 + x_2^2 - a^2\right) + x_1^2\,. \tag{3.1}$$

b) Let the boundary function be given as $Q_2(\boldsymbol{x}) = x_1^3$. On $\mathcal{S}_a(0,0)$ the boundary function reads

$$\widetilde{Q}_3(\varphi) = a^3 \cos^3\varphi \overset{(2.4)}{=} \frac{a^3}{4}\left(3\cos\varphi + \cos 3\varphi\right),$$

and the solution to the Dirichlet problem (2.1) in polar coordinates is evident

$$\begin{aligned}\mathring{u}(r,\varphi) &= \frac{3}{4}\,a^3\left(\frac{r}{a}\right)^1\cos\varphi + \frac{1}{4}\,a^3\left(\frac{r}{a}\right)^3\cos 3\varphi \overset{(2.6)}{=}\\ &= \frac{3}{4}\,a^2 r\cos\varphi + \frac{1}{4}\,r^3\left(\cos^3\!\varphi - 3\cos\varphi\sin^2\!\varphi\right).\end{aligned}$$

It readily follows from the latter that the solution in cartesian coordinates is

$$u(\boldsymbol{x}) = \frac{3}{4}\,a^2 x_1 + \frac{1}{4}\left(x_1^3 - 3\,x_1 x_2^2\right) = -\frac{3}{4}\,x_1\left(x_1^2 + x_2^2 - a^2\right) + x_1^3\,. \tag{3.2}$$

c) Let the boundary function be given as $Q_2(\boldsymbol{x}) = x_1^4$. On $\mathcal{S}_a(0,0)$ the boundary function reads

$$\widetilde{Q}_4(\varphi) = a^4\cos^4\!\varphi \overset{(2.5)}{=} \frac{a^4}{8}\left(3 + 4\,\cos 2\varphi + \cos 4\varphi\right),$$

and the solution to the Dirichlet problem (2.1) in polar coordinates is easily written as follows

$$\begin{aligned}\mathring{u}(r,\varphi) &= \frac{3}{8}\,a^4 + \frac{4}{8}\,a^4\left(\frac{r}{a}\right)^2\cos 2\varphi + \frac{1}{8}\,a^4\left(\frac{r}{a}\right)^4\cos 4\varphi =\\ &= \frac{3}{8}\left(a^4 - r^4\right) + \frac{4}{8}\left(a^2r^2 - r^4\right)\cos 2\varphi + \frac{1}{8}\left(r^4 - r^4\right)\cos 4\varphi + \mathring{Q}_4(r,\varphi) \overset{(2.6)}{=}\\ &= \frac{3}{8}\left(a^2 - r^2\right)\left(a^2 + r^2\right) + \frac{4}{8}\,r^2\left(a^2r^2 - r^2\right)\left(\cos^2\!\varphi - \sin^2\!\varphi\right) + \mathring{Q}_4(r,\varphi)\,.\end{aligned}$$

The latter can be rewritten in cartesian coordinates as follows

$$u(\boldsymbol{x}) = \left(x_1^2 + x_2^2 - a^2\right)\left(-\frac{3}{8}\,a^2 - \frac{7}{8}\,x_1^2 + \frac{1}{8}\,x_2^2\right) + x_1^4\,. \tag{3.3}$$

Note that representation (3.3) is obtained using the identical transformation of solution in polar coordinates, boundary function $\mathring{Q}_4(r,\varphi)$ being explicitly separated. ▲

Example 3.2. Consider the Dirichlet problem (1.9), where the boundary of the circle is specified by the equation

$$F_2(\boldsymbol{x}) = (x_1+2)^2 + (x_2-1)^2 - 9 = x_1^2 + x_2^2 + 4x_1 - 2x_2 - 4 = 0\,.$$

a) Let the boundary function be given as $Q_2(\boldsymbol{x}) = x_1 x_2$. On $\mathcal{S}_3(-2,1)$ according to formulas (1.3) the boundary function reads

$$\widetilde{Q}_2(\varphi) = -2 + 3\cos\varphi - 6\sin\varphi + \frac{9}{2}\sin 2\varphi\,,$$

and it is straightforward to write the solution to the Dirichlet problem (1.9) in polar coordinates

$$\mathring{u}(r,\varphi) = -2 + r\cos\varphi - 2\,r\sin\varphi + \frac{1}{2}\,r^2\sin 2\varphi\,.$$

The solution to the Dirichlet problem (1.9) in cartesian coordinates admits trivial representation (1.10)

$$u(\boldsymbol{x}) = x_1 x_2 = F_2(\boldsymbol{x})\,P_0(\boldsymbol{x}) + Q_2(\boldsymbol{x}) = Q_2(\boldsymbol{x})\,, \tag{3.4}$$

where polynomial of zero degree $P_0(\boldsymbol{x})$ equals zero. This conclusion is evident, since the boundary function $Q_2(\boldsymbol{x}) = x_1 x_2$ is a harmonic monomial (as it was noticed in Section 1).

b) Let the boundary function be given as $Q_3(\boldsymbol{x}) = x_1^2\,x_2$. On $\mathcal{S}_3(-2,1)$ the boundary function reads

$$\widetilde{Q}_3(\varphi) = +\frac{17}{2} - 12\cos\varphi + \frac{75}{4}\sin\varphi + \frac{9}{2}\cos 2\varphi - 18\sin 2\varphi + \frac{27}{4}\sin 3\varphi\,,$$

and using straightforward procedure the solution to the Dirichlet problem (1.9) is written as

$$\mathring{u}(r,\varphi) = +\frac{17}{2} - 4\,r\cos\varphi + \frac{25}{4}\,r\sin\varphi + \frac{1}{2}\,r^2\cos 2\varphi - 2\,r^2\sin 2\varphi + \frac{1}{4}\,r^3\sin 3\varphi\,.$$

When changing independent variables in function $\mathring{u}(r,\varphi)$ above from polar to cartesian coordinates one gets the solution to the Dirichlet problem (1.9) as a polynomial of third degree in variables x_1, x_2

$$u(\boldsymbol{x}) = +\frac{3}{4}\,x_1^2 x_2 - \frac{1}{4}\,x_2^3 - \frac{1}{4}\,x_1^2 - x_1 x_2 + \frac{1}{4}\,x_2^2 - x_1 + \frac{3}{2}\,x_2\,. \tag{3.5}$$

Representation (1.10) for the solution (3.5) is obtained using division with remainder $Q_3(\boldsymbol{x})$ of polynomial $u(\boldsymbol{x})$ by polynomial $F_2(\boldsymbol{x})$

$$u(\boldsymbol{x}) = \left(x_1^2 + x_2^2 + 4x_1 - 2x_2 - 4\right)\left(-\frac{1}{4}\,x_2 - \frac{1}{4}\right) + x_1^2 x_2\,. \tag{3.6}$$

c) Let the boundary function be given as $Q_4(\boldsymbol{x}) = x_1^2\,x_2^2$. On $\mathcal{S}_3(-2,1)$ the boundary function reads

$$\begin{aligned}\widetilde{Q}_4(\varphi) \;=\; &+\frac{293}{8} - 39\cos 1\varphi + \frac{75}{2}\sin 1\varphi - \frac{27}{2}\cos 2\varphi - 36\sin 2\varphi + \\ &+ 27\cos 3\varphi + \frac{27}{2}\sin 3\varphi \;-\; \frac{81}{8}\cos 4\varphi\,,\end{aligned}$$

and the solution to the Dirichlet problem (1.9) in polar coordinates is as follows

$$\begin{aligned}\mathring{u}(r,\varphi) \;=\; &+\frac{293}{8} - 13\,r\cos 1\varphi + \frac{25}{2}\,r\sin 1\varphi - \frac{3}{2}\,r^2\cos 2\varphi - 4\,r^2\sin 2\varphi + \\ &+ r^3\cos 3\varphi + \frac{1}{2}\,r^3\sin 3\varphi - \frac{1}{8}\,r^4\cos 4\varphi\,.\end{aligned}$$

Changing polar coordinates to cartesian ones in function $\mathring{u}(r,\varphi)$ above gives the solution to the Dirichlet problem (1.9) as a polynomial of fourth degree in variables x_1, x_2

$$u(\boldsymbol{x}) = -\frac{1}{8}\,x_1^4 + \frac{3}{4}\,x_1^2x_2^2 - \frac{1}{8}\,x_2^4 + \frac{3}{4}\,x_1^2 - 2\,x_1x_2 - \frac{3}{4}\,x_2^2 - 9\,x_1 + \frac{9}{2}\,x_2 + 7\,. \quad (3.7)$$

Division with remainder $Q_4(\boldsymbol{x})$ of polynomial $u(\boldsymbol{x})$ by polynomial $F_2(\boldsymbol{x})$ leads to representation (1.10) for the solution (3.7)

$$u(\boldsymbol{x}) = F_2(\boldsymbol{x})\left(-\frac{1}{8}\,x_1^2 - \frac{1}{8}\,x_2^2 + \frac{1}{2}\,x_1 - \frac{1}{4}\,x_2 - \frac{7}{4}\right) + Q_4(\boldsymbol{x})\,. \qquad (3.8)$$

d) Let the boundary function be given as $Q_5(\boldsymbol{x}) = x_1^3\,x_2^2$. On $\mathcal{S}_3(-2,1)$ the boundary function reads (using formulas (1.3) when changing variables (x_1, x_2) to variables (r,φ) is shown explicitly)

$$\begin{aligned}\widetilde{Q}_5(\varphi) \;=\; &(r\cos\varphi - 2)^3(r\sin\varphi + 1)^2\big|_{r=3} = \\ =\; &-\frac{527}{4} + \frac{1341}{8}\cos 1\varphi - 129\sin 1\varphi + 9\cos 2\varphi + \frac{297}{2}\sin 2\varphi - \\ &-\frac{1431}{16}\cos 3\varphi - 81\sin 3\varphi + \frac{243}{4}\cos 4\varphi + \frac{81}{4}\sin 4\varphi - \\ &-\frac{243}{16}\cos 5\varphi\,,\end{aligned}$$

and it is straightforward to obtain the solution to the Dirichlet problem (1.9) in polar coordinates as

$$\begin{aligned}\mathring{u}(r,\varphi) = &-\frac{527}{4} + \frac{447}{8}\,r\cos\varphi - 43\,r\sin\varphi + r^2\cos 2\varphi + \frac{33}{2}\,r^2\sin 2\varphi - \\ &-\frac{53}{16}\,r^3\cos 3\varphi - 3\,r^3\sin 3\varphi + \frac{3}{4}\,r^4\cos 4\varphi + \frac{1}{4}\,r^4\sin 4\varphi - \\ &-\frac{1}{16}\,r^5\cos 5\varphi\,.\end{aligned}$$

Applying trigonometric formulas (2.6) to the solution above and changing polar coordinates to cartesian ones according to (1.3) gives the solution to the Dirichlet problem (1.9) as a polynomial of fifth degree in variables x_1, x_2

$$\left\{\begin{aligned}u(\boldsymbol{x}) = &-\frac{1}{16}\,x_1^5 + \frac{5}{8}\,x_1^3x_2^2 - \frac{5}{16}\,x_1x_2^4 + \\ &+\frac{1}{8}\,x_1^4 - \frac{1}{4}\,x_1^3x_2 - \frac{3}{4}\,x_1^2x_2^2 + \frac{1}{4}\,x_1x_2^3 + \frac{1}{8}\,x_2^4 - \\ &-\frac{3}{16}\,x_1^3 - \frac{3}{2}\,x_1^2x_2 + \frac{9}{16}\,x_1x_2^2 + \frac{1}{2}\,x_2^3 - \\ &-\frac{29}{8}\,x_1^2 + \frac{67}{8}\,x_1x_2 + \frac{29}{8}\,x_2^2 + \frac{121}{4}\,x_1 - \frac{57}{4}\,x_2 - \frac{45}{2}\,.\end{aligned}\right. \tag{3.9}$$

There formally remain to prove function (3.9) being the solution to the Dirichlet problem (1.9). This means that both the Laplace equation and the boundary condition hold. Finding the first and the second repeated partial derivatives of function $u(\boldsymbol{x})$ (3.9):

$$\begin{aligned}\frac{\partial u}{\partial x_1} = &-\frac{5}{16}\,x_1^4 + \frac{15}{8}\,x_1^2x_2^2 - \frac{5}{16}\,x_2^4 + \frac{1}{2}\,x_1^3 - \frac{3}{4}\,x_1^2x_2 - \frac{3}{2}\,x_1x_2^2 + \frac{1}{4}\,x_2^3 - \\ &-\frac{9}{16}\,x_1^2 + \frac{9}{16}\,x_2^2 - 3x_1x_2 - \frac{29}{4}\,x_1 + \frac{67}{8}\,x_2 + \frac{121}{4}\,, \\ \frac{\partial u}{\partial x_2} = &+\frac{5}{4}\,x_1^3x_2 - \frac{5}{4}\,x_1x_2^3 - \frac{1}{4}\,x_1^3 - \frac{3}{2}\,x_1^2x_2 + \frac{3}{4}\,x_1x_2^2 + \frac{1}{2}\,x_2^3 + \\ &+\frac{3}{2}\,x_2^2 + \frac{9}{8}\,x_1x_2 - \frac{3}{2}\,x_1^2 - \frac{67}{8}\,x_1 + \frac{29}{4}\,x_2 - \frac{57}{4}\,, \\ \frac{\partial^2 u}{\partial x_1^2} = &-\frac{5}{4}\,x_1^3 + \frac{15}{4}\,x_1x_2^2 + \frac{3}{2}\,x_1^2 - \frac{3}{2}\,x_1x_2 - \frac{3}{2}\,x_2^2 - \frac{9}{8}\,x_1 - 3x_2 - \frac{29}{4} = -\frac{\partial^2 u}{\partial x_2^2}\,,\end{aligned}$$

demonstrates that the Laplace equation actually holds.

To write down representation (1.10)

$$u(\boldsymbol{x}) = F_2(\boldsymbol{x})\, P_3(\boldsymbol{x}) + Q_5(\boldsymbol{x})\,, \tag{3.10}$$

one should perform division with remainder $Q_5(\boldsymbol{x})$ of polynomial $u(\boldsymbol{x})$ by polynomial $F_2(\boldsymbol{x})$ to obtain

$$P_3(\boldsymbol{x}) = -\frac{1}{16}\,x_1^3 - \frac{5}{16}\,x_1 x_2^2 + \frac{3}{8}\,x_1^2 - \frac{3}{8}\,x_1 x_2 + \frac{1}{8}\,x_2^2 - \frac{31}{16}\,x_1 + \frac{3}{4}\,x_2 + \frac{45}{8}\cdot \tag{3.11}$$

From representation (3.10), (3.11) for function $u(\boldsymbol{x})$ (3.9) there immediately follows that the boundary condition holds as well.

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