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INVARIANT APPROACH TO THE SOLVABILITY OF FEEDBACK DESIGN PROBLEM FOR LINEAR CONTROL SYSTEMS

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New necessary and sufficient solvability conditions of a linear output feedback design problem for linear control system are offered. With the help of got conditions a solution of robust regulator design problem for the linear control system is also shown. Examples are given.

Keywords: linear control system, output feedback, modal control problem, pole placement, exterior degree.

1. Introduction

Consider the linear autonomous control system whose state equation is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \mathbf{x}(t) \in \mathbb{R}^n, \mathbf{u}(t) \in \mathbb{R}^m, \tag{1.1}$$

and an output equation has the form

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \mathbf{y}(t) \in \mathbb{R}^p. \tag{1.2}$$

Here $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^p$ are real linear spaces of vector-columns of dimensionalities $n,\ m,\ p$; $\mathbf{x}(t) = (x_1(t), ..., x_n(t))^T$, $\mathbf{u}(t) = (u_1(t), ..., u_m(t))^T$, $\mathbf{y}(t) = (y_1(t), ..., y_p(t))^T$ are vectors of states, inputs, and outputs; $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{B} : \mathbb{R}^m \to \mathbb{R}^n$, $\mathbf{C} : \mathbb{R}^n \to \mathbb{R}^p$ are real linear maps of appropriate spaces.

Introduce into system (1.1) a linear feedback under the law

$$\mathbf{u}(t) = \mathbf{K}\mathbf{y}(t) = \mathbf{K}\mathbf{C}\mathbf{x}(t), \tag{1.3}$$

where $\mathbf{K} : \mathbb{R}^p \to \mathbb{R}^m$ is a real linear operator. Then we have the so-called closed-loop control system $\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{BKC})\mathbf{x}(t)$, whose properties completely are determined by properties of the operator $\mathbf{A} + \mathbf{BKC}$. If the operator $\mathbf{C}$ is irreversible, then feedback (1.3) is called an output feedback; in opposite case feedback (1.3) it is called a state feedback.

Fix any bases in spaces $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$; then the triple of operators $\mathbf{A}, \mathbf{B}, \mathbf{C}$ will be represented in the chosen bases by matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. Further, we will adhere only to these denotations.

Let $I_n \in \mathbb{R}^{n\times n}$ be the identity mapping. Denote by $f_i(K)$ the sum of all principal minors of order i of matrix $A + BKC$, where $1 \le i \le n$.

Feedback design problem. *It is necessary to construct a real matrix $K \in \mathbb{R}^{m\times p}$ such that the characteristic polynomial of the matrix $A + BKC$ would coincide with a desired real polynomial of degree n:*

$$\begin{aligned}\det(\lambda I_n - A - BKC) &\equiv \lambda^n - f_1(K)\lambda^{n-1} + ... + (-1)^n f_n(K)\\ &= \lambda^n + d_1\lambda^{n-1} + ... + d_n; \quad d_1, ..., d_n \in \mathbb{R}.\end{aligned}$$

Thus, the feedback design problem is reduced to research of the system of equations

$$-d_1 = f_1(K), d_2 = f_2(K), ..., (-1)^n d_n = f_n(K). \tag{1.4}$$

The feedback design problem of linear control laws for linear control systems (there are other names of this problem: *pole placement by static output feedback* or *modal control problem*) was first formulated in the strict statement by R. E. Kalman in his lecture at the 1th IFAC World Congress in 1960. The concepts of the controllability and observability were base concepts for this statement, which were first also introduced by R. E. Kalman at the same 1th IFAC World Congress. Importance of the controllability and observability conditions exist in that these conditions are necessary for solvability of the feedback design problem.

In our opinion, the whole development period of the modal control theory for finite-dimensional systems can be separated on four stages.

The first stage began approximately in 1960 (from the moment of the strict statement of this problem by R. E. Kalman) and continued to 1967, when W. M. Wonham [12] proved the theorem that if the state feedback control is used, then the complete controllability is a necessary and sufficient condition for the solvability of modal control problem; article [12] is a starting point for book [13], in which the author applied linear algebra tools to the solution of practical tasks of linear control theory.

The second stage (approximately 1968 – 1975) is already characterized by research of the output feedback design problem (the vector of states is inaccessible to direct observation). Here, the most essential result was given by H. Kimura [6], which proved that if $m + p > n$, then the pole placement problem for the closed-loop control system almost always has a real solution. The majority of articles for this period differ by a linear approach, which was actively used. (The basic idea of this linear approach consists in that the control system in the state space is transformed so that coefficients of the characteristic polynomial of the closed-loop system (or even their part) linearly depended on the coefficients of matrix K.)

The third development stage is a twelve-years period (approximately 1976 – 1987), which is characterized that L. R. Fletcher *et al.* [5, 7] found the necessary and sufficient solvability conditions of the modal control problem in the case $m + p > n$ and the following restriction: a spectrum of the closed-loop system contains only different eigenvalues.

Note that in the papers [5,7] it was actually shown that if condition $m+p > n$ is not fulfilled, then the feedback design problem becomes substantially nonlinear (the linear approach mentioned above in this case is not suitable) and for a solution of this problem it is necessary to apply completely other methods.

The use of these methods began in 1981 when the mathematicians R. W. Brockett and C. I. Byrnes [4] first essentially applied exterior algebra tools to the modal control problem. Here the solvability conditions of the modal control problem for the case $mp = n$ were found. Their paper was the exact beginning of the fourth stage, which had been used to the end of twentieth century. Among the large number of publications on applications of exterior algebra in the modal control theory, it should be noted the significance of X. Wang [10, 11], which for general linear systems proved the solvability of the feedback design problem in the case $mp \geq n$. However, in articles [4, 10, 11] of any recommendations for practical realization of these new ideas it were not offered. (An attempt of the development constructive procedure of solution of the feedback design problem in the case $mp = n$ was undertaken in [1]. In addition, the constructive procedure of solution of the feedback design problem in the case $mp > n$ was offered in [2].)

An important problem, which can arise up at the solution of feedback design problem, is a robustness problem of the got matrix K [2, 3, 9]. It is clear that for research of the robust properties of this matrix methods using transformations of coordinates become practically useless. (Essentially, the robustness problem is the design problem of matrix K for system (1.1), (1.2) depending on some parameters: $A(\Theta), B(\Theta), C(\Theta)$, where Θ is a parameter vector.) Therefore, it is desirable to derive such equations for the feedback design problem solution, which will not depend on transformations of coordinates in the space state. These equations we will call invariant.

2. Equations of feedback design problem

Without loss of generality we can consider that the matrix A in system (1.1) is reduced to the diagonal form

$$A \to S^{-1}AS = \mathtt{diag}(\lambda_1, ..., \lambda_n)$$

by the invertible transformation $S \in \mathbb{C}^{n\times n}$. (In this case the matrices B and C in equations (1.1), (1.2) will be transformed to matrices: $B \to S^{-1}B$ and $C \to CS$.)

Assume that $B = (b_1, ..., b_m)$ $(C = (c_1^T, ..., c_p^T)^T)$ are columns (rows) of the matrix B (C). Let also l be an integer such that $1 \leq l \leq n$. Denote by $C^{\overline{j_1\cdots j_l}} \in \mathbb{C}^{p\times(n-l)}$ $(B_{\overline{j_1\cdots j_l}} \in \mathbb{C}^{(n-l)\times m})$ a matrix obtained from the matrix C (the matrix B) in which columns (rows) with the indexes $1 \leq j_1 < ... < j_l \leq n$ are missed.

Let $a_1, ..., a_n$ be the coefficients of characteristic polynomial $\det(\lambda I_n - A)$ of the matrix A.

Theorem 2.1. *Let* $A = \mathtt{diag}(\lambda_1, ..., \lambda_n)$. *Then equations of feedback design problem for system (1.1), (1.2) have the form:*

$$\begin{cases} d_1 - a_1 = -(\wedge^1(CB))_h \cdot (\wedge^1 K)^h, \\ d_2 - a_2 = \Big(\sum\limits_{1\le i_1\le n} \lambda_{i_1} \cdot \wedge^1(C^{\overline{i_1}} B_{\overline{i_1}})\Big)_h \cdot (\wedge^1 K)^h + (\wedge^2(CB))_h \cdot (\wedge^2 K)^h, \\ d_3 - a_3 = -\Big(\sum\limits_{1\le i_1<i_2\le n} \lambda_{i_1}\lambda_{i_2} \cdot \wedge^1(C^{\overline{i_1 i_2}} B_{\overline{i_1 i_2}})\Big)_h \cdot (\wedge^1 K)^h - \\ \qquad \Big(\sum\limits_{1\le i_1\le n} \lambda_{i_1} \cdot \wedge^2(C^{\overline{i_1}} B_{\overline{i_1}})\Big)_h \cdot (\wedge^2 K)^h - (\wedge^3(CB))_h \cdot (\wedge^3 K)^h, \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots, \\ d_n - a_n = \\ \quad (-1)^n \Big(\sum\limits_{1\le i_1<\ldots<i_{n-1}\le n} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_{n-1}} \cdot \wedge^1(C^{\overline{i_1\ldots i_{n-1}}} B_{\overline{i_1\ldots i_{n-1}}})\Big)_h \cdot (\wedge^1 K)^h + \\ \quad (-1)^n \Big(\sum\limits_{1\le i_1<\ldots<i_{n-2}\le n} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_{n-2}} \cdot \wedge^2(C^{\overline{i_1\ldots i_{n-2}}} B_{\overline{i_1\ldots i_{n-2}}})\Big)_h \cdot (\wedge^2 K)^h + \\ \ldots + (-1)^n \Big(\sum\limits_{1\le i_1<\ldots<i_{n-p}\le n} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_{n-p}} \cdot \wedge^p(C^{\overline{i_1\ldots i_{n-p}}} B_{\overline{i_1\ldots i_{n-p}}})\Big)_h \cdot (\wedge^p K)^h. \end{cases} \tag{2.1}$$

Proof. From the first equation of system (1.4) we have:

$$-d_1 = f_1(K) = \operatorname{tr} A + \operatorname{tr}\left((b_1, ..., b_m) \cdot \begin{pmatrix} k_{11} & \cdots & k_{1p} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mp} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}\right) =$$

$$\operatorname{tr} A + \operatorname{tr}\left(\Big(\sum_{i=1}^m b_i k_{i1}, ..., \sum_{i=1}^m b_i k_{ip}\Big) \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}\right) =$$

$$\operatorname{tr} A + \operatorname{tr}\Big(c_{11}\sum_{i=1}^m b_i k_{i1} + ... + c_{p1}\sum_{i=1}^m b_i k_{ip}, ..., c_{1n}\sum_{i=1}^m b_i k_{i1} + ... + c_{pn}\sum_{i=1}^m b_i k_{ip}\Big) =$$

$$\operatorname{tr} A + \operatorname{tr}\Big(\sum_{j=1}^p c_{j1}\sum_{i=1}^m b_{1i}k_{ij} + \sum_{j=2}^p c_{j2}\sum_{i=1}^m b_{2i}k_{ij} + ... + \sum_{j=1}^p c_{jn}\sum_{i=1}^m b_{ni}k_{ij}\Big) =$$

$$\operatorname{tr} A + \operatorname{tr}\sum_{j=1}^p\sum_{i=1}^m\Big(\sum_{l=1}^n (c_{jl}b_{li})k_{ij}\Big) =$$

$$\operatorname{tr} A + (c_1 b_1, ..., c_1 b_m, ..., c_p b_1, ..., c_p b_m) \cdot (\wedge^1 K)^h = -a_1 + (\wedge^1(CB))_h \cdot (\wedge^1 K)^h.$$

Now we represent the matrix $A + BKC$ in the following form:

$$\begin{pmatrix} \lambda_1 + \sum_{i=1}^p c_{i1}(\sum_{j=1}^m b_{1j}k_{ji}) & \ldots & \sum_{i=1}^p c_{in}(\sum_{j=1}^m b_{1j}k_{ji}) \\ \sum_{i=1}^p c_{i1}(\sum_{j=1}^m b_{2j}k_{ji}) & \ldots & \sum_{i=1}^p c_{in}(\sum_{j=1}^m b_{2j}k_{ji}) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^p c_{i1}(\sum_{j=1}^m b_{nj}k_{ji}) & \ldots & \lambda_n + \sum_{i=1}^p c_{in}(\sum_{j=1}^m b_{nj}k_{ji}) \end{pmatrix}. \tag{2.2}$$

Then from the second equation of system (1.4) and representation (2.2) it follows that

$$f_2(K)=\sum_{1\le j_1<j_2\le n}\Bigg(\lambda_{j_1}\lambda_{j_2}+\lambda_{j_1}\sum_{i=1}^{p}c_{ij_2}(\sum_{j=1}^{m}b_{j_2j}k_{ji})+\lambda_{j_2}\sum_{i=1}^{p}c_{ij_1}(\sum_{j=1}^{m}b_{j_1j}k_{ji})+$$

$$\Big(\sum_{i=1}^{p}c_{ij_1}(\sum_{j=1}^{m}b_{j_1j}k_{ji})\Big)\Big(\sum_{i=1}^{p}c_{ij_2}(\sum_{j=1}^{m}b_{j_2j}k_{ji})\Big)-$$

$$-\Big(\sum_{i=1}^{p}c_{ij_2}(\sum_{j=1}^{m}b_{j_1j}k_{ji})\Big)\Big(\sum_{i=1}^{p}c_{ij_1}(\sum_{j=1}^{m}b_{j_2}k_{ji})\Big)\Bigg)=$$

$$a_2+\Big(\sum_{1\le i_1\le n}\lambda_{i_1}\cdot\wedge^1(C^{\overline{i_1}}B_{\overline{i_1}})\Big)_h\cdot(\wedge^1K)^h+$$

$$\sum_{1\le i_1<i_2\le p}\Big(\sum_{1\le j_1<j_2\le n}(c_{i_1j_1}c_{i_2j_2}-c_{i_1j_2}c_{i_2j_1})\cdot$$

$$((\sum_{j=1}^{m}b_{j_1j}k_{ji_1})(\sum_{j=1}^{m}b_{j_2j}k_{ji_2})-(\sum_{j=1}^{m}b_{j_1j}k_{ji_2})(\sum_{j=1}^{m}b_{j_2j}k_{ji_1}))\Big)=$$

$$a_2+\Big(\sum_{1\le i_1\le n}\lambda_{i_1}\cdot\wedge^1(C^{\overline{i_1}}B_{\overline{i_1}})\Big)_h\cdot(\wedge^1K)^h+$$

$$\sum_{1\le i_1<i_2\le p}\sum_{1\le j_1<j_2\le n}\det\begin{pmatrix}c_{i_1j_1}&c_{i_1j_2}\\ c_{i_2j_1}&c_{i_2j_2}\end{pmatrix}\cdot\det\begin{pmatrix}\sum_{j=1}^{m}b_{j_1j}k_{ji_1}&\sum_{j=1}^{m}b_{j_1j}k_{ji_2}\\ \sum_{j=1}^{m}b_{j_2j}k_{ji_1}&\sum_{j=1}^{m}b_{j_2j}k_{ji_2}\end{pmatrix}=$$

$$a_2+\Big(\sum_{1\le i_1\le n}\lambda_{i_1}\cdot\wedge^1(C^{\overline{i_1}}B_{\overline{i_1}})\Big)_h\cdot(\wedge^1K)^h+(\wedge^2(CB))_h\cdot(\wedge^2K)^h.$$

Now starting from an concept of induction on $l=1,...,n$, it is easily to get a general formula from representation (2.2) of the matrix $A+BKC$:

$$d_l-a_l=(-1)^l\Big(\sum_{1\le i_1<...<i_{l-1}\le n}\lambda_{i_1}\cdot...\cdot\lambda_{i_{l-1}}\cdot\wedge^1(C^{\overline{i_1...i_{l-1}}}B_{\overline{i_1...i_{l-1}}})\Big)_h\cdot(\wedge^1K)^h+$$

$$(-1)^l\Big(\sum_{1\le i_1<...<i_{l-2}\le n}\lambda_{i_1}\cdot...\cdot\lambda_{i_{l-2}}\cdot\wedge^1(C^{\overline{i_1...i_{l-2}}}B_{\overline{i_1...i_{l-2}}})\Big)_h\cdot(\wedge^2K)^h+...+$$

$$(-1)^l\Big(\sum_{1\le i_1<...<i_{l-p}\le n}\lambda_{i_1}\cdot...\cdot\lambda_{i_{l-p}}\cdot\wedge^1(C^{\overline{i_1...i_{l-p}}}B_{\overline{i_1...i_{l-p}}})\Big)_h\cdot(\wedge^pK)^h.$$

□

3. Invariant equations of feedback design problem for $\min(m, p) = 2$

Further, we will consider that $p = 2$ and $m \geq p$. (If $m \leq p$, then we introduce renames $A^T \to A,\ C^T \to B,\ B^T \to C$. As a result we obtain desired inequality: $m \geq p$.) Let $K = (k_{ij}) \in \mathbb{R}^{m \times 2}$ be the matrix of output feedback and let k_1, k_2 be columns of this matrix. Similarly, let $Q = (q_{ij}) \in \mathbb{R}^{2 \times m}$ be the matrix with rows q_1, q_2. Denote by $k_1 \wedge k_2$ the column

$$\wedge^2 K = \wedge^2(k_1, k_2) = (k_{11}k_{22} - k_{12}k_{21}, ..., k_{m-1,1}k_{m2} - k_{m-1,2}k_{m1})^T \in \mathbb{R}^{m(m-1)/2}$$

and denote by $q_1 \wedge q_2$ the row

$$\wedge^2 Q = \wedge^2 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = (q_{11}q_{22} - q_{12}q_{21}, ..., q_{1,m-1}q_{2,m} - q_{1,m}q_{2,m-1}) \in \mathbb{R}^{m(m-1)/2}.$$

Denote by c_1 and c_2 the rows of matrix C in chosen bases of the spaces $\mathbb{R}^n$ and $\mathbb{R}^2$.

Theorem 3.1. *The invariant system equations of feedback design problem is such:*

$$\begin{cases} d_1 = a_1 - (c_1 B, c_2 B) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \\ d_2 = a_2 - (c_1(A + a_1 I_n)B, c_2(A + a_1 I_n)B) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1 B \\ c_2 B \end{pmatrix} (k_1 \wedge k_2), \\ \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ , \\ d_n = a_n - (c_1(\sum\limits_{j=0}^{n-1} a_j A^{n-j-1})B, c_2(\sum\limits_{j=0}^{n-1} a_j A^{n-j-1})B) \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \\ \qquad \left(\sum\limits_{j=0}^{n-2} a_j \left(\sum\limits_{i=0}^{n-j-2} \wedge^2 \begin{pmatrix} c_1 A^i B \\ c_2 A^{n-i-j-2} B \end{pmatrix} \right) \right) (k_1 \wedge k_2), \end{cases} \tag{3.1}$$

where $a_0 = 1$ and $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ is a column of dimension $2m$, which composed from columns k_1 and k_2.

Proof. We take advantage of Theorem (2.1). Denote by $C^{j_1 \cdots j_l} \in \mathbb{C}^{p \times l}$ ($B_{j_1 \cdots j_l} \in \mathbb{C}^{l \times m}$) a matrix composed from the columns of matrix C (the rows of matrix B) with the indexes $1 \leq j_1 < ... < j_l \leq n$.

We specify on obvious property of the matrices $C^{j_1 \cdots j_l}$ and $B_{j_1 \cdots j_l}$:

$$C^{j_1 \cdots j_l} B_{j_1 \cdots j_l} = C^{j_1} B_{j_1} + ... + C^{j_l} B_{j_l} \in \mathbb{C}^{p \times m}.$$

In the second equation of system (2.1), we transform the following sum:

$$\sum_{1 \leq i_1 \leq n} \lambda_{i_1} \cdot \wedge^1 (C^{\overline{i_1}} B_{\overline{i_1}}) =$$

$$\sum_{1\le i_1\le n}\lambda_{i_1}\cdot\wedge^1(CB-C^{i_1}B_{i_1})=-a_1CB-\sum_{1\le i_1\le n}\lambda_{i_1}\cdot\wedge^1(C^{i_1}B_{i_1})=$$

$$-a_1CB-C\cdot\begin{pmatrix}\lambda_1&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&\lambda_n\end{pmatrix}\cdot B=-a_1CB-CAB.$$

In the third equation of system (2.1), we transform the following sum:

$$\sum_{1\le i_1<i_2\le n}\lambda_{i_1}\lambda_{i_2}\cdot\wedge^1(C^{\overline{i_1i_2}}B_{\overline{i_1i_2}})=\sum_{1\le i_1<i_2\le n}\lambda_{i_1}\lambda_{i_2}\cdot\wedge^1(CB-C^{i_1i_2}B_{i_1i_2})=$$

$$a_2CB-\sum_{1\le i_1<i_2\le n}\lambda_{i_1}\lambda_{i_2}\cdot\wedge^1C^{i_1i_2}B_{i_1i_2}=$$

$$=a_2CB-\sum_{1\le i_1<i_2\le n}\lambda_{i_1}\lambda_{i_2}\cdot(\wedge^1C^{i_1}B_{i_1}+\wedge^1C^{i_2}B_{i_2})=a_2CB-$$

$$-\Big[(\lambda_1+\lambda_2+...+\lambda_n)\cdot(\lambda_1\wedge^1C^1B_1+\lambda_2\wedge^1C^2B_2+...+\lambda_n\wedge^1C^nB_n)+$$

$$(\lambda_1+\lambda_2+...+\lambda_n)\cdot(\lambda_1\wedge^1C^1B_1+\lambda_2\wedge^1C^2B_2+...+\lambda_n\wedge^1C^nB_n)+...+$$

$$(\lambda_1+\lambda_2+...+\lambda_n)\cdot(\lambda_1\wedge^1C^1B_1+\lambda_2\wedge^1C^2B_2+...+\lambda_n\wedge^1C^nB_n)-$$

$$(\lambda_1^2\wedge^1C^1B_1+\lambda_2^2\wedge^1C^2B_2+...+\lambda_n^2\wedge^1C^nB_n)\Big]=$$

$$a_2CB+a_1C\cdot\begin{pmatrix}\lambda_1&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&\lambda_n\end{pmatrix}\cdot B+C\cdot\begin{pmatrix}\lambda_1^2&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&\lambda_n^2\end{pmatrix}\cdot B=$$

$$=a_2CB+a_1CAB+CA^2B.$$

Now in the third equation of system (2.1), we transform another sum:

$$\sum_{1\le i_1\le n}\lambda_{i_1}\cdot\wedge^2(C^{\overline{i_1}}B_{\overline{i_1}})=$$

$$=\sum_{1\le i_1\le n}\lambda_{i_1}\cdot\wedge^2(CB)-\sum_{1\le i_1\le n}\lambda_{i_1}\sum_{1\le i_2\le n,i_1\ne i_2}\wedge^2(C^{i_1i_2}B_{i_1i_2})=$$

$$-a_1\wedge^2(CB)-(\lambda_1+\lambda_2)\wedge^2(C^{12}B_{12})-(\lambda_1+\lambda_3)\wedge^2(C^{13}B_{13})-...-$$

$$(\lambda_{n-1}+\lambda_n)\wedge^2(C^{n-1,n}B_{n-1,n})=-a_1\wedge^2(CB)-$$

$$-\wedge^2\left(\begin{pmatrix}\lambda_1c_{11},\lambda_2c_{12}\\ c_{21},c_{22}\end{pmatrix}\cdot B_{12}\right)-\wedge^2\left(\begin{pmatrix}c_{11},c_{12}\\ \lambda_1c_{21},\lambda_2c_{22}\end{pmatrix}\cdot B_{12}\right)-$$

$$\wedge^2\left(\begin{pmatrix}\lambda_1c_{11},\lambda_3c_{13}\\ c_{21},c_{23}\end{pmatrix}\cdot B_{13}\right)-\wedge^2\left(\begin{pmatrix}c_{11},c_{13}\\ \lambda_1c_{21},\lambda_3c_{23}\end{pmatrix}\cdot B_{13}\right)-...-$$

$$\wedge^2\left(\begin{pmatrix} \lambda_{n-1}c_{1,n-1}, \lambda_n c_{1n} \\ c_{2,n-1}, c_{2,n} \end{pmatrix} \cdot B_{n-1,n}\right) - \wedge^2\left(\begin{pmatrix} c_{1,n-1}, c_{1,n} \\ \lambda_{n-1}c_{2,n-1}, \lambda_n c_{2,n} \end{pmatrix} \cdot B_{n-1,n}\right) =$$

$$-a_1 \wedge^2 (CB) - \wedge^2 \begin{pmatrix} (\lambda_1 c_{11}, ..., \lambda_n c_{1n})B \\ c_2 B \end{pmatrix} - \wedge^2 \begin{pmatrix} c_1 B \\ (\lambda_1 c_{21}, ..., \lambda_n c_{2n})B \end{pmatrix} =$$

$$-a_1 \wedge^2 (CB) - \wedge^2 \begin{pmatrix} c_1 AB \\ c_2 B \end{pmatrix} - \wedge^2 \begin{pmatrix} c_1 B \\ c_2 AB \end{pmatrix}.$$

In the fourth equation of system (2.1), we transform the following sum:

$$\sum_{1\le i_1<i_2\le n} \lambda_{i_1}\lambda_{i_2} \cdot \wedge^2 (C^{\overline{i_1 i_2}} B_{\overline{i_1 i_2}}) =$$

$$\sum_{1\le i_1<i_2\le n} \lambda_{i_1}\lambda_{i_2} \cdot \wedge^2 \begin{pmatrix} c_1 B - c_1^{i_1} B_{i_1} - c_1^{i_2} B_{i_2} \\ c_2 B - c_2^{i_1} B_{i_1} - c_2^{i_2} B_{i_2} \end{pmatrix} = \sum_{1\le i_1<i_2\le n} \lambda_{i_1}\lambda_{i_2} \cdot \wedge^2 \begin{pmatrix} c_1 B \\ c_2 B \end{pmatrix} +$$

$$\sum_{1\le i_1<i_2\le n} \left(\wedge^2 \begin{pmatrix} \lambda_{i_1} c_1^{i_1} B_{i_1} \\ \lambda_{i_2} c_2^{i_2} B_{i_2} \end{pmatrix} + \wedge^2 \begin{pmatrix} \lambda_{i_2} c_1^{i_2} B_{i_1} \\ \lambda_{i_1} c_2^{i_1} B_{i_2} \end{pmatrix}\right) -$$

$$\sum_{1\le i_1<i_2\le n} \lambda_{i_1}\lambda_{i_2} \left(\wedge^2 \begin{pmatrix} c_1 B \\ c_2^{i_1} B_{i_1} + c_2^{i_2} B_{i_2} \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1^{i_1} B_{i_1} + c_1^{i_2} B_{i_2} \\ c_2 B \end{pmatrix}\right) =$$

$$a_2 \wedge^2 \begin{pmatrix} c_1 B \\ c_2 B \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1 AB \\ c_2 AB \end{pmatrix} -$$

$$\sum_{1\le i_1<i_2\le n} \lambda_{i_1}\lambda_{i_2} \left(\wedge^2 \begin{pmatrix} c_1 B \\ c_2^{i_1} B_{i_1} \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1 B \\ c_2^{i_2} B_{i_2} \end{pmatrix} +\right.$$

$$\left. + \wedge^2 \begin{pmatrix} c_1^{i_1} B_{i_1} \\ c_2 B \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1^{i_2} B_{i_2} \\ c_2 B \end{pmatrix}\right) +$$

$$\sum_{1\le i_1\le n} \left(\wedge^2 \begin{pmatrix} \lambda_{i_1}^2 c_1^{i_1} B_{i_1} \\ c_2 B \end{pmatrix}\right) + \sum_{1\le i_2\le n} \left(\wedge^2 \begin{pmatrix} c_1 B \\ \lambda_{i_2}^2 c_2^{i_2} B_{i_2} \end{pmatrix}\right) =$$

$$a_2 \wedge^2 \begin{pmatrix} c_1 B \\ c_2 B \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1 AB \\ c_2 AB \end{pmatrix} +$$

$$a_1 \left(\wedge^2 \begin{pmatrix} c_1 B \\ c_2 AB \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1 AB \\ c_2 B \end{pmatrix}\right) + \wedge^2 \begin{pmatrix} c_1 A^2 B \\ c_2 B \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1 B \\ c_2 A^2 B \end{pmatrix}.$$

Now starting from an concept of induction on $l = 1, ..., n$, it is easily to get the general representation (3.1) of the invariant system equations of feedback design problem for arbitrary matrix $A \in \mathbb{R}^{n\times n}$.

The proof of Theorem (3.1) was is done in supposition that the matrix A has different eigenvalues.

Let W_n be the subset of all matrices in $\mathbb{R}^{n\times n}$ such that any matrix from W_n has different eigenvalues. It is known that W_n is open and dense in $\mathbb{R}^{n\times n}$ [8]. Let $f(T)$ and $g(T)$ be two regular polynomial functions depending on an arbitrary matrix $T \in \mathbb{R}^{n\times n}$. If $\forall T \in W_n$ we have $f(T) = g(T)$, then $\forall T \in \mathbb{R}^{n\times n}$ we also have $f(T) = g(T)$. Obviously, that right parts of equations (3.1) can be considered as the regular polynomial functions depending on A. From here it follows that Theorem (3.1) is valid for any matrix $A \in \mathbb{R}^{n\times n}$. □

Corollary 3.1. *The invariant system equations of output feedback design problem may be transformed to the following form:*

$$\begin{cases} f_1 = -(c_1B, c_2B)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \\ f_2 = -(c_1AB, c_2AB)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \wedge^2\begin{pmatrix} c_1B \\ c_2B \end{pmatrix}(k_1\wedge k_2), \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ f_n = -(c_1A^{n-1}B, c_2A^{n-1}B)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \left(\sum_{i=0}^{n-2}\wedge^2\begin{pmatrix} c_1A^iB \\ c_2A^{n-i-2}B \end{pmatrix}\right)(k_1\wedge k_2), \end{cases} \tag{3.2}$$

where $f_1, ..., f_n$ are arbitrary real numbers.

Proof. Introduce the following product of $(n \times n)$-matrices:

$$P(a_1, ..., a_{n-1}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & -a_1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & -a_1 & 1 & 0 \\ 0 & \cdots & -a_2 & 0 & 1 \end{pmatrix} \cdot \ldots \cdot$$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -a_{n-2} & 0 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then system (3.2) can be got from system (3.1) by the multiplication of both parts of system (3.1) on the matrix $P(a_1, ..., a_{n-1})$. Here $(f_1, ..., f_n)^T = P(a_1, ..., a_{n-1})^T \cdot (d_1 - a_1, ..., d_n - a_n)^T$. □

Introduce the matrices:

$$G = \begin{pmatrix} c_1B & c_2B \\ c_1AB & c_2AB \\ c_1A^2B & c_2A^2B \\ c_1A^3B & c_2A^3B \\ \ldots & \ldots \\ c_1A^{n-1}B & c_2A^{n-1}B \end{pmatrix} \in \mathbb{R}^{n\times 2m},$$

$$H = \begin{pmatrix} 0 \\ \wedge^2 \begin{pmatrix} c_1 B \\ c_2 B \end{pmatrix} \\ \wedge^2 \begin{pmatrix} c_1 B \\ c_2 AB \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1 AB \\ c_2 B \end{pmatrix} \\ \wedge^2 \begin{pmatrix} c_1 B \\ c_2 A^2 B \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1 AB \\ c_2 AB \end{pmatrix} + \wedge^2 \begin{pmatrix} c_1 A^2 B \\ c_2 B \end{pmatrix} \\ \cdots\cdots \\ \sum_{i=0}^{n-2} \wedge^2 \begin{pmatrix} c_1 A^i B \\ c_2 A^{n-i-2} B \end{pmatrix} \end{pmatrix} \in \mathbb{R}^{n \times m(m-1)/2}.$$

Let $f = (f_1, ..., f_n)^T \in \mathbb{R}^n$. Then system (3.2) may be rewritten as

$$f = -G \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + H \cdot (k_1 \wedge k_2). \tag{3.3}$$

The following theorem is a trivial corollary of the known result of algebraic geometry [8].

Theorem 3.2. *Let f be an arbitrary real vector in system (3.3). In order that there existed a complex matrix $K = (k_1, k_2) \in \mathbb{C}^{m \times 2}$ satisfying to system (3.3) (K is a function of f) it is necessary and sufficient that*

$$\operatorname{rank}(G, H) = n.$$

It means that if we define the map $\phi : K \to (d_1, ..., d_n)$ depending on matrices A, B, and C by formula (1.4), then this map must be surjective: $\phi(\mathbb{C}^{m \times p}) = \mathbb{C}^n$.

It is known that in order that the map $\phi(K)$ there was surjective its necessary that the rank of Jacobi matrix $\partial\phi(K)/\partial K$ at $K = 0$ was equal to n for all points $(X, Y, Z) \in \Sigma$ from some open set $\Sigma \subset \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{p \times n}$. In the case $\min(m, p) = 2$ the set Σ is defined by the conditions of Theorem (3.2).

It should be said that Theorem (3.2) does not solve the feedback design problem (the matrix K have to be real). However, this theorem can appear useful in two following cases.

1. Theorem (3.2) gives an obvious solvability criterion of the feedback design problem (lets even in the complex case). (Note that conditions of complete controllability and observability of system (1.1), (1.2) (see [2], [4–7], [10–13]) are not sufficient for solvability of the feedback design problem!)

2. Theorems (3.1) and (3.2) allow to investigate robust properties of the matrix K.

Indeed, let the matrices $A(\Theta)$, $B(\Theta)$, and $C(\Theta)$ in system (1.1), (1.2) depend on some real parametric vector $\Theta \in \Omega$, where $\Omega \subset \mathbb{R}^k$ is some open bounded set. Then we derive system (3.3), in which $f = f(\Theta)$, $G = G(\Theta)$, and $H = H(\Theta)$ are explicit functions of Θ.

Let $\Theta_0 \in \Omega$ be some fixed vector. With the help of the methods, which were presented in [2,3], and [9], we find the feedback matrix $K = K(\Theta_0)$. Now already it is possible to estimate a radius of sphere $\mathbb{S} := \{\|\Theta - \Theta_0\| \leq r\}$ such that $\forall\Theta \in \mathbb{S}$ the matrix $A(\Theta)+B(\Theta)K(\Theta_0)C(\Theta)$ is stable. The robust properties of the matrix $K(\Theta_0)$ so much the better than more the magnitude of radius r.

4. Examples

1. Let $m = p = 2$ and $n = 4$. Consider the system

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 \\ -2 & 2 & 1 & -1 \\ -2 & 4 & 1 & 1 \\ 0 & 1 & -1 & -3 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 3 & 1 \\ -2 & 1 \\ -1 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ 13 \\ 50 \end{pmatrix} + \begin{pmatrix} 6 & 1 & 1 & -1 \\ 4 & 9 & 8 & -6 \\ 23 & -12 & -61 & 43 \\ 23 & -62 & -340 & 10 \end{pmatrix} \cdot \begin{pmatrix} k_{11} \\ k_{21} \\ k_{12} \\ k_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ -7 \\ -64 \\ -417 \end{pmatrix} \cdot \det K.$$

Here $\operatorname{rank} G = \operatorname{rank}(G, H) = 4$. Thus, the conditions of Theorem (3.2) are fulfilled.

Assume that $d_1 = 10$, $d_2 = 35$, $d_3 = 50$, $d_1 = 24$. (It means that the roots of characteristic polynomial of closed system are $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3, \lambda_4 = -4$.)

In this case we have two feedback matrices:

$$K_1 = \begin{pmatrix} 1.1539 & -0.0970 \\ 3.5772 & -0.5963 \end{pmatrix}, K_2 = \begin{pmatrix} 1.2864 & 2.1699 \\ 0.3067 & -0.8047 \end{pmatrix}.$$

2. Let again $m = p = 2$ and $n = 4$. Consider the system

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of matrix A is $\lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1$. Thus, we have $a_1 = -4, a_2 = 6, a_3 = -4, a_4 = 1$.

For this system we derive

$$\operatorname{rank}(B, AB, A^2B, A^3B) = \operatorname{rank}(C^T, (A^T)C^T, (A^T)^2C^T, (A^T)^3C^T) = 4.$$

Therefore, this system is complete controllable and observable.

Nevertheless, we have

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 3 & 0 & 0 & 3 \\ 4 & 0 & 0 & 4 \end{pmatrix}, \ H = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 10 \end{pmatrix},$$

and $\operatorname{rank} G = 1$, $\operatorname{rank}(G, H) = 2$. Thus, the conditions of Theorem (3.2) are not fulfilled. From here it follows that for the last system the feedback design problem is insolvable.

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